A Rate of Convergence for Loop-Erased Random Walk

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Outline

The Problem

In "Conformal invariance of planar loop-erased random walks and uniform spanning trees" (AOP, 2004), Lawler, Schramm, and Werner showed that the scaling limit of loop-erased random walk is a process called SLE_2 (Schramm-Loewner Evolution with parameter 2). The proof is qualitative and no rate of convergence immediately follows from it.

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Loop-Erased Random Walk

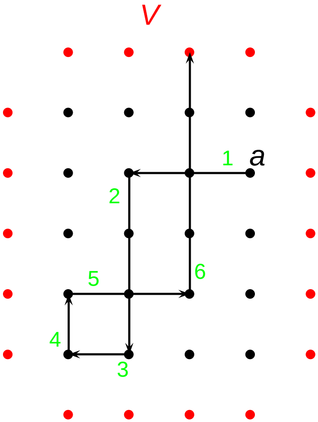
Consider a connected graph $G \subseteq \mathbb{Z}^2$, a vertex $a \in G$, and a nonempty set $V \subset G$. Loop-erased random walk (LERW) γ from a to V is defined as follows:

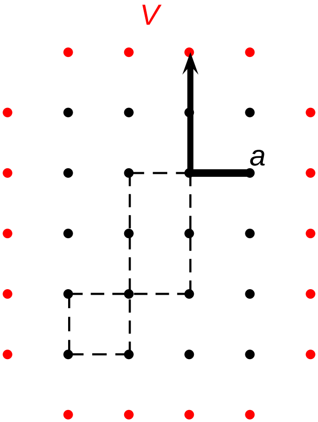
Let $\{S(n)\}_{n \geq 0}$ be simple random walk on G started at a and $T_V = \inf\{n \geq 0 : S(n) \in V\}$. $\gamma = (\gamma_0, \ldots, \gamma_\ell)$ is defined inductively by

- $\gamma_0 = a$,
- for $n \geq 0$,
  - if $\gamma_n \in V, n = \ell$,
  - if $\gamma_n \notin V, \gamma_{n+1} = S(k+1)$, where $k = \max\{m \leq T_V : S(m) = \gamma_n\}$.

For $a, b \in \mathbb{Z}^2$, the loop-erasure of S from a to b and of its time-reversal are not usually the same (path by path). However, they have the same distribution.

LERW from a to V





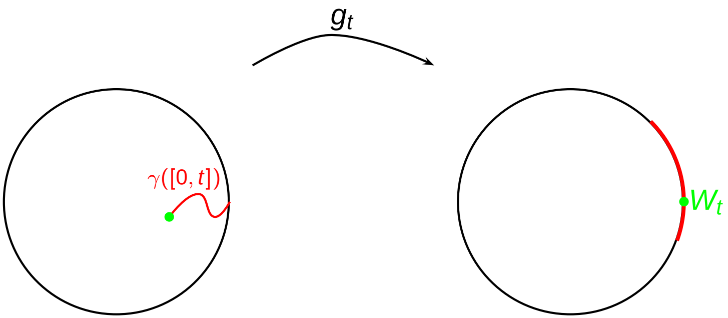
The Loewner Equation

- Consider a simple curve $\gamma : [0, \infty] \to \bar{\mathbb{U}} = \{z \in \mathbb{C} : |z| \leq 1\}$ in the unit disk going from the boundary to the origin, i.e., $\gamma(0) \in \partial\mathbb{U}$ and $\gamma(\infty) = 0$.
- By the Riemann mapping theorem, for all $s \geq 0$, there is a unique conformal map $g_s : \mathbb{U} \setminus \gamma(0, s] \to \mathbb{U}$ such that $g_s(0) = 0, g_s'(0) > 0$.
- g_s' is increasing in s, so we can reparametrize γ by capacity t so that $g_t'(0) = e^t$.
- The maps g_t satisfy the Loewner equation

$$\partial_t g_t(z) = g_t(z)\frac{W(t) + g_t(z)}{W(t) - g_t(z)}, \qquad g_0(z) = z,$$

where $W(t) = \lim_{z \to \gamma(t)} g_t(z)$. W is the driving function of the curve γ.

The Loewner Map



The Schramm-Loewner Evolution (SLE)

Instead of obtaining a driving function W from a simple curve γ, we can also start with the driving function W and obtain a family of maps $g_\cdot$ from it via Loewner's equation.

If $W : [0, \infty) \to \partial\mathbb{U}$ is continuous, then for every $z \in \bar{\mathbb{U}}$, there is a solution $g_t(z)$ to the Loewner (O)DE up to some time $\tau(z)$. If we let $K_t = \{z \in \bar{\mathbb{U}} : \tau(z) \leq t\}, g_t$ is then defined precisely on $\mathbb{U} \setminus K_t$.

The (radial) Schramm-Loewner Evolution (SLE) is defined to be the process $(K_t)_{t\geq 0}$ obtained from Loewner's equation when using $W(t) = e^{iB_{\kappa t}}$, where B is a standard Brownian motion:

$$\partial g_t(z) = -g_t(z)\frac{g_t(z) + e^{iB_{\kappa t}}}{g_t(z) - e^{iB_{\kappa t}}}, \qquad g_0(z) = z.$$

The SLE Curve

Definition

The *SLE curve (or trace)* is $\gamma(t) = g_t^{-1}(W(t)), t \geq 0$.

Theorem (Rohde-Schramm, 2001)

With probability one,

- $0 \leq \kappa \leq 4$: $\gamma(t)$ is a random, simple curve avoiding the unit circle.
- $4 < \kappa < 8$: $\gamma(t)$ is not a simple curve. It has double points, but does not cross itself! These paths do hit the unit circle.
- $\kappa \geq 8$: $\gamma(t)$ is a space filling curve. It has double points, but does not cross itself. Yet it is space-filling...

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Theorem (Beffara, 2008)

With probability one, the Hausdorff dimension of the SLE_κ trace is

$$\min\left\{1+\frac{\kappa}{8}, 2\right\}.$$

Other SLE's

In general simply connected domains D, we can define radial SLE from $w \in \partial D$ to $z \in D$ as the conformal image of radial SLE in the unit disk under the map $\psi : \mathbb{U} \mapsto D$ with $\psi(1) = w$ and $\psi(0) = z$.

There is a half-plane version of SLE, going from 0 to ∞ in $\{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$, called chordal SLE. It satisfies

$$\partial g_t(z) = \frac{2}{g_t(z) - B_{\kappa t}}, \qquad g_0(z) = z.$$

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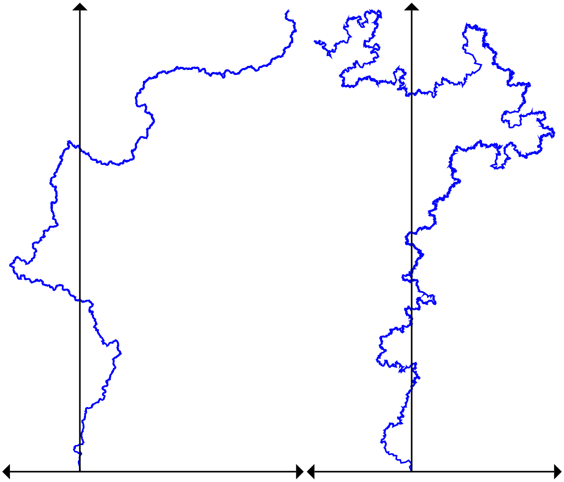

$\kappa = 1$ $\kappa = 2$

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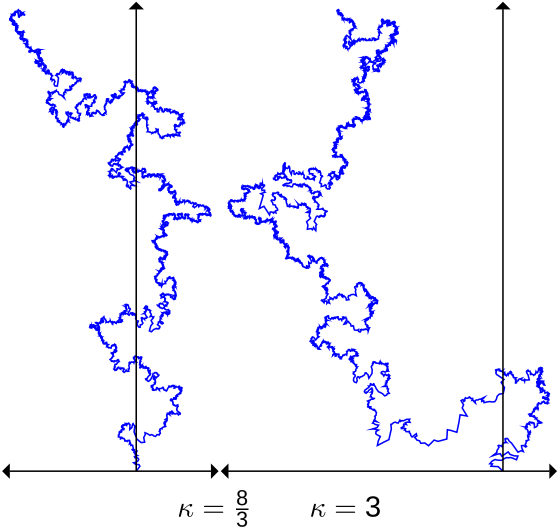


The Success Story of SLE

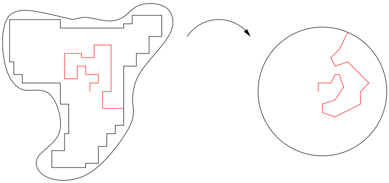
- Application to planar Brownian motion:

$$dim_H(\text{frontier}) = 4/3, \quad dim_H(\text{cutpoints}) = 3/4.$$

- SLE_2 is the scaling limit of LERW.
- SLE_3 is the scaling limit of the critical Ising model interface.
- SLE_4 is the scaling limit of the harmonic explorer and the discrete Gaussian free field interface.
- SLE_6 is the scaling limit of the critical percolation exploration path on the triangular lattice.
- SLE_8 is the scaling limit of the uniform spanning tree Peano curve.
- $SLE_{8/3}$ is **conjectured** to be the scaling limit of the self-avoiding walk.

From LERW to SLE

Setting:



- Let $D \ni 0$ be a simply connected planar domain with $\frac{1}{n}\mathbb{Z}^2$ grid domain approximation $D_n \subset \mathbb{C}$. (A grid domain D is a domain whose boundary is a union of edges of the scaled lattice.)
- $\psi_{D_n} : D_n \to \mathbb{D}$, $\psi_{D_n}(0) = 0, \psi'_{D_n}(0) > 0$.
- γ_n: time-reversed LERW from 0 to ∂D_n (on $\frac{1}{n}\mathbb{Z}^2$).
- $\tilde{\gamma}_n = \psi_{D_n}(\gamma_n)$ is a path in $\mathbb{D}$. Parametrize by capacity.
- $W_n(t) = W_0 e^{i\vartheta_n(t)}$: the Loewner driving function for $\tilde{\gamma}_n$.

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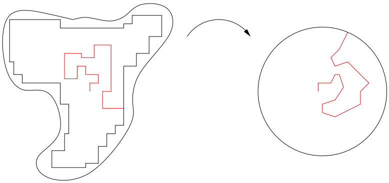

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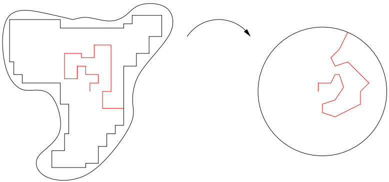

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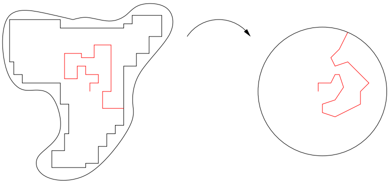

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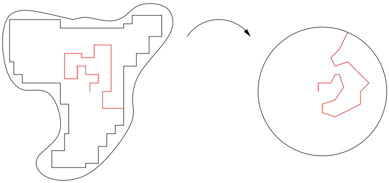

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LSW's result

A weak form of the convergence result is the following:

Theorem (Lawler, Schramm, Werner, 2004)

Let $\mathcal{D}$ be the set of simply connected grid domains with $0 \in D, D \neq \mathbb{C}$. For every $T > 0, \epsilon > 0$, there exists $n = n(T, \epsilon)$ such that if $D \in \mathcal{D}$ has inner radius $> n$, then there exists a coupling between loop-erased random walk γ from ∂D to 0 in D and Brownian motion B started uniformly on $[0, 2\pi]$ such that

$$P(\sup\{|\theta(t) - B_{2t}| : t \in [0, T] > \epsilon\} < \epsilon,$$

where $\theta(t)$ satisfies $W(t) = W(0)e^{i\theta(t)}$ and $W(t)$ is the driving process of γ in Loewner's equation.

This result leads to the stronger convergence of paths with respect to Hausdorff metric.

Ideas of Proof

Three main steps:

1. Find a discrete martingale observable for the LERW path. Prove that it converges to something conformally invariant involving $\gamma(t)$.
2. Use Step 1 together with the Loewner equation to show that the Loewner driving function is almost a martingale with the right (conditional) variance.
3. Use Step 2 and Skorokhod embedding to couple the Loewner driving function with a Brownian motion and show that they are uniformly close with high probability.

To obtain a rate we re-examined these steps to find explicit bounds on error terms.

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A Martingale Observable

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Introduction
The Problem
The Protagonists
Loop-Erased Random Walk
An Eight-Slide Introduction to SLE

Convergence of LERW to SLE_2

A Rate of Convergence
Step 1: An Observable
A Green's Function Estimate
Step 2: The Driving Function
Step 3: From Driving Function to Brownian Motion

The martingale observable used by Lawler-Schramm-Werner is the discrete Poisson kernel:

Fix $n \in \mathbb{N}$ and $z \in V(D_n) := D_n \cap \mathbb{Z}^2$ and let

$$M_k = M_k(z) := \frac{H_k(z, \gamma_n(k))}{H_k(0, \gamma_n(k))}, \quad k \geq 0,$$

where for $x \in V(D_n) \setminus \gamma_n[0, k]$, $H_k(x, \gamma_n(k))$ is the probability that simple random walk started at x exits the slit domain $D_n \setminus \gamma_n[0, k]$ at $\gamma_n(k)$, i.e., discrete harmonic measure.

One can show that M_k is a martingale with respect to $\gamma_n[0, k]$ and for fixed k, $M_k(\cdot)$ is discrete harmonic.

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Discrete and Continuous Poisson Kernels

The next step is to show that for appropriate z when n is large, the discrete and continuous Poisson kernels are close:

$$\frac{H_k(z, \gamma_n(k))}{H_k(0, \gamma_n(k))} \approx \frac{1 - |\psi_k(z)|^2}{|\psi_k(z) - \psi_k(\gamma_n(k))|^2}$$

with explicit error terms (in terms of the lattice scale $1/n$).
Here $\psi_k : D_n \setminus \gamma_n[0, k] \to \mathbb{D}$.

Kozdron-Lawler have a similar estimate in *Estimates of Random Walk Exit Probabilities and Application to LERW*. However, it is in a slightly different setting (union of squares domains) and isn't optimal.

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Discrete and Continuous Poisson Kernels

Theorem (B-Johansson-Kozdron, 2010)

Let $0 < \epsilon < 1/4$ and let $0 < \rho < 1$ be fixed. Suppose that D is a grid domain with inrad$(D) \geq n$. Furthermore, suppose that $x \in D \cap \mathbb{Z}^2$ with $|\psi_D(x)| \leq \rho$ and $u \in V_\partial(D)$. If both x and u are accessible by a simple random walk starting from 0, then

$$\frac{H_D(x,u)}{H_D(0,u)} = \frac{1 - |\psi_D(x)|^2}{|\psi_D(x) - \psi_D(u)|^2} \cdot [\, 1 + O(n^{-(1/4-\epsilon)})\,].$$

Proving the Closeness of the Poisson Kernels

The proof relies on the fact that for all $\epsilon > 0$, we can find $\delta > 0$ such that if

- D is a(n appropriate) grid domain,
- $E \subset \partial D$ is a union of edges of $\mathbb{Z}^2$,
- x is far enough from ∂D,

then

$$H(x) \geq \epsilon \Rightarrow h(x) \geq \delta, \qquad (*)$$

where $h(x) = h_D(x, E)$ is the discrete harmonic measure of E at x and $H(x)$ is its continuous analogue.

The implication in (*) is generally not satisfied by grid domains (problems arise with fjords or channels).

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Proving the Closeness of the Poisson Kernels

To get around this problem, one can cut off anything in the domain that isn't accessible by random walk started at x by creating a Union of Big Squares (UBS) domain D_0, which is the union of squares of side length 2 centered at the points of $V_0(D)$.

Beurling's theorem implies that if the Poisson kernels are close in D_0, they are close in D: If $\mathrm{inrad}(D) \geq n$,

$$\partial\psi_D(D_0) \subset \mathcal{A}(1 - cn^{-1/2}, 1).$$

The conformal map from $\partial\psi(D_0)$ to $\mathbb{D}$ is almost the identity and one can show (writing ψ_0 for ψ_{D_0}) that

$$|\psi_0(x)| = |\psi_D(x)| + \mathcal{O}\left(n^{-1/2}\log n\right)$$

and

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Proving the Closeness of the Poisson Kernels

To get around this problem, one can cut off anything in the domain that isn't accessible by random walk started at x by creating a Union of Big Squares (UBS) domain D_0, which is the union of squares of side length 2 centered at the points of $V_0(D)$.

Beurling's theorem implies that if the Poisson kernels are close in D_0, they are close in D: If $\mathrm{inrad}(D) \geq n$,

$$\partial\psi_D(D_0) \subset \mathcal{A}(1 - cn^{-1/2}, 1).$$

The conformal map from $\partial\psi(D_0)$ to $\mathbb{D}$ is almost the identity and one can show (writing ψ_0 for ψ_{D_0}) that

$$|\psi_0(x)| = |\psi_D(x)| + \mathcal{O}\left(n^{-1/2}\log n\right)$$

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Suppose that we know

$$\frac{H_{D_0}(x,u)}{H_{D_0}(0,u)} = \frac{1-|\psi_0(x)|^2}{|\psi_0(x)-\psi_0(u)|^2} \cdot [\,1+O(n^{-(1/4-\epsilon)})\,].$$

Then

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Proving the Closeness of the Poisson Kernels

$$H_{D_0}(z,w) = \frac{1}{4}\sum_{y\sim w} G_{D_0}(z,y),$$

so $\frac{H_{D_0}(x,u)}{H_{D_0}(0,u)} = \frac{1-|\psi_0(x)|^2}{|\psi_0(x)-\psi_0(u)|^2} \cdot [\,1+O(n^{-(1/4-\epsilon)})\,]$ follows (somewhat directly) from

Theorem

For $0<\epsilon<1/4$ and $0<\rho<1$, if D is UBS with $\mathrm{inrad}(D)=n$, $x,y\in D\cap\mathbb{Z}^2$ with $|\psi_D(x)|\le\rho$ and $|\psi_D(y)|\ge 1-n^{-(1/4-\epsilon)}$, then

$$\frac{G_D(x,y)}{G_D(y)} = \frac{1-|\psi_D(x)|^2}{|\psi_D(x)-e^{i\theta_D(y)}|^2}\cdot[\,1+O(n^{-(1/4-\epsilon)})\,]$$

where G_D is Green's function for SRW on $D\cap\mathbb{Z}^2$.

An Estimate for Exiting the Domain

Since

$$g_D(z) = E^z[\log |B_{T_D}|] - \log |z|$$

and

$$G_D(x) = E^x[a(S_{\tau_D})] - a(x),$$

where $a(x) = \frac{2}{\pi} \log |x| + k_0 + O(|x|^{-2})$, a key computation for comparing Poisson kernels is:

Proposition

For any $\epsilon > 0$ there exists $c > 0$ such that if D is a UBS domain with $\mathrm{inrad}(D) = n$, then for $|x| \leq n^2$,

$$|E^x[\log |B_T|] - E^x[\log |S_\tau|]| \leq cn^{-(1/2-\epsilon)},$$

where $T = \inf\{t \geq 0 : B_t \notin D\}, \tau = \inf\{k \geq 0 : S(k) \notin D\}$.

KMT Approximation

To prove this proposition, we couple B and S using the KMT approximation:

Theorem

There exist a probability space containing a planar standard Brownian motion and a two-dimensional simple random walk and a constant $c > 0$ such that for every $\lambda > 0$, every $n \in \mathbb{N}$,

$$P\Big(\sup_{0\leq t\leq n} |S_{2t} - B_t| > c(\lambda+1)\log n\Big) \leq cn^{-\lambda}.$$

An Estimate for Exiting the Domain

A number of configurations can make $\left|\log|B_T| - \log|S_\tau|\right|$ unusually large, for instance if:

- we are on one of the rare events of KMT.
- T is very large, meaning that KMT has "lost strength"
- $|B_T|$ and $|S_\tau|$ are very large

An Estimate for Exiting the Domain

Suppose the starting point x is 0. We define

$$\mathcal{A}_k = \{|B_T| \in [n^{1+k\epsilon}, n^{1+(k+1)\epsilon})\}, k \geq 0$$
$$\mathcal{B}_\ell = \{T \in [n^{\ell\epsilon}, n^{(\ell+1)\epsilon})\}, \ell \geq 0$$
$$\mathcal{C}_m = \{|B_T - S_\tau| \in [n^{m\epsilon}, n^{(m+1)\epsilon})\}, m \geq 0$$
$$\mathcal{H}_r = \{\sup_{0 \leq t \leq T} |B_t - S_t| \in [cr \log T, c(r+1) \log T)\}, r \geq 0,$$

and

$$\mathcal{E}_{k,\ell,m,r} = \mathcal{A}_k \cap \mathcal{B}_\ell \cap \mathcal{C}_m \cap \mathcal{H}_r.$$

Then, obviously,

$$|E[\log|B_T|] - E[\log|S_\tau|]| \leq \sum_{k,\ell,m,r \geq 0} E[|\log|B_T| - \log|S_\tau||\mathbb{1}\{\mathcal{E}_{k,\ell,m,r}\}]. \quad (1)$$

Technical Ingredients

Lemma

If B is a planar Brownian motion, S a planar simple random walk, there exists a constant $C > 0$ such that for every $n \geq 0$, every $r \geq 1$,

$$P(\sup_{0 \leq t \leq n} |B(t)| \geq r\sqrt{n}) \leq C \exp\left\{-r^2/2\right\}, \tag{2}$$

$$P(\max_{0 \leq k \leq 2n} |S(k)| \geq r\sqrt{n}) \leq C \exp\left\{-r^2/4\right\}. \tag{3}$$

Technical Ingredients

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planar simple random walk, there exists a constant $C > 0$ *such that for every* $n \geq 0$*, every* $r \geq 1$,

$$P(\sup_{0 \leq t \leq n} |B(t)| \leq r^{-1}\sqrt{n}) \leq \exp\left\{-Cr^2\right\}, \tag{4}$$

$$P(\max_{0 \leq k \leq 2n} |S(k)| \leq r^{-1}\sqrt{n}) \leq \exp\left\{-Cr^2\right\}.$$

Technical Ingredients

Lemma (Beurling Estimate)

There exists a constant $c > 0$ such that for any $R \geq 1$, any $x \in \mathbb{C}$ with $|x| \leq R$, any $A \subset \mathbb{C}$ with $[0, R] \subset \{|z| : z \in A\}$,

$$P^x(\xi_R \leq T_A) \leq c\,(|x|/R)^{1/2}\,, \tag{5}$$

where $\xi_R = \inf\{t \geq 0 : |B(t)| \geq R\}$ and $T_A = \inf\{t \geq 0 : B(t) \in A\}$, where B is planar Brownian motion.

An Estimate for Exiting the Domain

The sum

$$\sum_{k,\ell,m,r\geq 0} E[\big|\log|B_T|-\log|S_\tau|\big|\mathbb{1}\{\mathcal{E}_{k,\ell,m,r}\}]$$

is difficult to evaluate, partly because under the KMT coupling, B and S are not jointly Markov. However, the dominant term in the sum is easy to detect heuristically:

$$|\log|B_T|-\log|S_\tau|| = O(\log(1+\frac{|B_T-S_\tau|}{|B_T|})) = O(\frac{|B_T-S_\tau|}{|B_T|}).$$

Most likely, $|B_T|\approx n$. On this event, for $a\leq 1, P(|B_T-S_\tau|\approx n^a)\leq n^{-a/2}$, by Beurling. The contribution of this event to the sum is therefore

$$n^{-a/2}n^a/n = n^{a/2-1}.$$

This is maximal when $a=1$. The cases $a>1$ all yield a smaller contribution to the sum.

Completing the Main Estimate

Theorem

For any $\epsilon > 0$ there exists $c > 0$ such that if D is a UBS domain with $\text{inrad}(D) = n$, then for $|x| \leq n^2$,

$$|E^x[\log|B_T|] - E^x[\log|S_\tau|]| \leq cn^{-(1/2-\epsilon)},$$

where $T = \inf\{t \geq 0 : B_t \notin D\}, \tau = \inf\{k \geq 0 : S(k) \notin D\}$.

This theorem gives bad estimates for Green's functions if x is close to ∂D. To complete the proof of our Poisson kernel estimate, a number of technical results based on the distortion theorem, Koebe theorem, Beurling estimates, etc. were needed.

The needed Tools from Complex Analysis

Theorem

Let D be a simply connected domain and suppose $f : D \to \mathbb{C}$ is a conformal map. Set $d = d(z, \partial D)$ for $z \in D$. If $|z - w| \le rd$, then

$$\frac{1-r}{(1+r)^3}|f'(z)| \le |f'(w)| \le \frac{1+r}{(1-r)^3}|f'(z)|,$$

$$\frac{|f'(z)|}{(1+r)^2}|z-w| \le |f(z)-f(w)| \le \frac{|f'(z)|}{(1-r)^2}|z-w|,$$

and

$$\mathcal{B}(f(z), d|f'(z)|/4) \subset f(D),$$

where $\mathcal{B}(w, \rho)$ denotes the open disk of radius ρ around w.

Finding the Mean and Variance of the Driving Function

We now apply

$$\frac{H_D(x,u)}{H_D(0,u)} = \frac{1-|\psi_D(x)|^2}{|\psi_D(x)-\psi_D(u)|^2}\left[1+O\left(\frac{n^{-1/4}\log n}{|\psi_D(x)-\psi_D(u)|}\right)\right]$$

to the domains $D_n \setminus \gamma_n[0,k]$:

Choose $k = k(n)$ so that t_k, the capacity of $\gamma_n[0,k]$ is on an intermediate scale (of order $n^{-1/6}$).

Fix an appropriate $z \in V(D_n)$ and set

$$\lambda_k := \frac{1-|\psi_k(z)|^2}{|\psi_k(z)-\psi_k(\gamma_n(k))|^2} = \frac{\mathrm{Re}(\psi_k(\gamma_n(k)))+\mathrm{Re}(\psi_k(z))}{\mathrm{Re}(\psi_k(\gamma_n(k)))-\mathrm{Re}(\psi_k(z))}.$$

This is almost a martingale with respect to $\gamma_n[0,j]$.

We can express λ_k in terms of ψ_0, the Loewner equation, t_k and $\vartheta(t_k) = \psi_k(\gamma_n(k))$.

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Finding the Mean and Variance of the Driving Function

We can Taylor-expand $\lambda_k - \lambda_0$, take expectations and compare coefficients; the fact that λ_k is almost a martingale implies (after some work involving the Beurling and distortion estimates) that

$$\mathbb{E}[\vartheta(t_k)] = \mathcal{O}\left(n^{-1/4} \log n\right)$$

and

$$\mathbb{E}[\vartheta(t_k)^2 - 2t_k] = \mathcal{O}\left(n^{-1/4} \log n\right).$$

Coupling the Driving Function with Brownian Motion

Grow a macroscopic piece of the curve (capacity of order 1), pieced together by $\approx n^{1/6}$ intermediate scale pieces of the curve ($\gamma([t_{m_{k-1}}, t_{m_k}])$).

The pieces correspond to increasing times/capacities t_{m_k}. From Step 2, $k \mapsto \vartheta(t_{m_k})$ is almost, though not quite, a martingale (with small increments). However,

$$\xi_j = \vartheta(t_{m_j}) - \vartheta(t_{m_{j-1}}) - E[\vartheta(t_{m_j}) - \vartheta(t_{m_{j-1}}) | \gamma[0, t_{m_{j-1}}]]$$

is a martingale difference sequence and

$$M_k = \sum_{j=0}^{k} \xi_j$$

can be embedded into Brownian motion.

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Coupling the Driving Function with Brownian Motion

Lemma (Skorokhod embedding theorem)

If $(M_n)_{n \le N}$ is an $(\mathcal{F}_n)_{n \le N}$ martingale, with $\|M_n - M_{n-1}\|_\infty \le 2\,\delta$ and $M_0 = 0$ a.s., then there are stopping times $0 = \tau_0 \le \tau_1 \le \cdots \le \tau_N$ for standard Brownian motion $(B_t, t \ge 0)$, such that $(M_0, M_1, \ldots, M_N)$ and $(B(\tau_0), B(\tau_1), \ldots, B(\tau_N))$ have the same law. Moreover, one can impose for $n = 0, 1, \ldots, N-1$

$$\mathbb{E}\left[\tau_{n+1} - \tau_n \,|\, B[0, \tau_n]\right] = \mathbb{E}\left[(B(\tau_{n+1}) - B(\tau_n))^2 \,|\, B[0, \tau_n]\right],$$

$$\mathbb{E}\left[(\tau_{n+1} - \tau_n)^p \,|\, B[0, \tau_n]\right] \le C_p \mathbb{E}\left[(B(\tau_{n+1}) - B(\tau_n))^{2p} \,|\, B[0, \tau_n]\right]$$

for constants $C_p < \infty$, and also

$$\tau_{n+1} \le \inf\left\{t \ge \tau_n : \left|B_t - B_{\tau_n}\right| \ge 2\,\delta\right\}.$$

Coupling the Driving Function with Brownian Motion

We get stopping times τ_j such that:

$$B(\tau_j) \approx \vartheta(t_{m_j}).$$

To show that the "Brownian motion time" is close to $2\times$ capacity time, i.e., $\tau_j \approx 2t_{m_j}$, we consider the natural "time" associated to M:

$$Y_k := \sum_{j=1}^{k} \xi_j^2, \quad k = 1, \ldots, K.$$

and first show that Y_k is close to $2t_{m_k}$, using a martingale maximal inequality due to Haeusler and the fact that

$$\mathbb{E}[\vartheta(t_k)^2 - 2t_k] \approx 0.$$

Haeusler's Inequality

Lemma

Let ξ_k, $k = 1, \ldots, K$, be a martingale difference sequence with respect to the filtration $\mathcal{F}_k$. Then for all $\lambda, u, v > 0$

$$P\left(\max_{1\le j\le K} |\sum_{k=1}^{j} \xi_k| \ge \lambda\right) \le \sum_{k=1}^{K} P(|\xi_k| > u)$$
$$+ 2P\left(\sum_{k=1}^{K} E(\xi_k^2|\mathcal{F}_{k-1}) > v\right)$$
$$+ \exp\{\lambda u^{-1}(1 - \log(\lambda u v^{-1}))\}.$$

A Rate

All of this gives

Theorem (B-J-K, 2010)

Let $0 < \epsilon < 1/24$ be fixed, and let D be a simply connected domain with inner radius 1. For every $T > 0$ there exists an $n_0 < \infty$ depending only on T such that whenever $n > n_0$ there is a coupling of γ^n with Brownian motion B_t, $t \geq 0$, where e^{iB_0} is uniformly distributed on the unit circle, with the property that

$$P\left(\sup_{0\leq t\leq T} |W_n(t) - e^{iB_{2t}}| > n^{-(1/24-\epsilon)}\right) < n^{-(1/24-\epsilon)}.$$

A Rate

Transferring (nontrivially) an estimate for the chordal SLE map close to the tip to the radial case also gives

Theorem

Let $0 < t \leq t_0$ where t_0 is small enough. There exists $c < \infty$ such that for n sufficiently large there is a coupling of LERW $\tilde{\gamma}^n$ with SLE_2 $\tilde{\gamma}$ such that if $p < (15 - 8\sqrt{3})/66$,

$$P\left(d_H\left(\tilde{\gamma}^n[0,t] \cup \partial\mathbb{D}, \tilde{\gamma}[0,t] \cup \partial\mathbb{D}\right) > c(\log n)^{-p}\right) < c(\log n)^{-p}.$$

Here, for two compact sets A, $B \subset \mathbb{C}$, $d_H(A,B) = \inf\left\{\epsilon > 0 : A \subset \bigcup_{z\in B} D(z,\epsilon),\ B \subset \bigcup_{z\in A} D(z,\epsilon)\right\}$ denotes Hausdorff distance.