APPENDIX 1 TO CHAPTER 4

Models of Asset Pricing

In this appendix, we first examine why diversification, the holding of many risky assets in a portfolio, reduces the overall risk an investor faces. Then we will see how diversification affects the price of assets by developing models of asset pricing.

Benefits of Diversification

Our discussion of the theory of asset demand indicates that most people like to avoid risk; that is, they are risk-averse. Why, then, do most investors hold many risky assets rather than just one? Doesn't holding many risky assets expose the investor to more risk?

The old warning about not putting all your eggs in one basket holds the key to the answer: Because holding many risky assets (called *diversification*) reduces the overall risk an investor faces, diversification is beneficial. To see why this is so, let's look at some specific examples of how an investor fares on his investments when he is holding two risky securities.

Consider two assets: common stock of Frivolous Luxuries, Inc., and common stock of Bad Times Products, Unlimited. When the economy is strong, which we'll assume is one-half of the time, Frivolous Luxuries has high sales and the return on the stock is 15%; when the economy is weak, the other half of the time, sales are low and the return on the stock is 5%. On the other hand, suppose that Bad Times Products thrives when the economy is weak, so that its stock has a return of 15%, but it earns less when the economy is strong and has a return on the stock of 5%. Since both these stocks have an expected return of 15% half the time and 5% the other half of the time, both have an expected return of 10%. However, both stocks carry a fair amount of risk, because there is uncertainty about their actual returns.

Suppose, however, that instead of buying one stock or the other, Irving the Investor puts half his savings in Frivolous Luxuries stock and the other half in Bad Times Products stock. When the economy is strong, Frivolous Luxuries stock has a return of 15%, while Bad Times Products has a return of 5%. The result is that Irving earns a return of 10% (the average of 5% and 15%) on his holdings of the two stocks. When the economy is weak, Frivolous Luxuries has a return of only 5% and Bad Times Products has a return of 15%, so Irving still earns a return of 10% regardless of whether the economy is strong or weak. Irving is better off from this strategy of diversification because his expected return is 10%, the same as from holding either Frivolous Luxuries or Bad Times Products alone, and yet he is not exposed to *any* risk.

Although the case we have described demonstrates the benefits of diversification, it is somewhat unrealistic. It is quite hard to find two securities with the characteristic that when the return of one is high, the return of the other is always low.[1] In the real world, we are more likely to find at best returns on securities that are independent of each other; that is, when one is high, the other is just as likely to be high as to be low.

Suppose that both securities have an expected return of 10%, with a return of 5% half the time and 15% the other half of the time. Sometimes both securities will earn the higher return and sometimes both will earn the lower return. In this case if Irving holds equal amounts of each security, he will on average earn the same return as if he had just put all his savings into one of these securities. However, because the returns on these two securities are independent, it is just as likely that when one earns the high 15% return, the other earns the low 5% return and vice versa, giving Irving a return of 10% (equal to the expected return). Because Irving is more likely to earn what he expected to earn when he holds both securities instead of just one, we can see that Irving has again reduced his risk through diversification.[2]

The one case in which Irving will not benefit from diversification occurs when the returns on the two securities move perfectly together. In this case, when the first security has a return of 15%, the other also has a return of 15% and holding both securities results in a return of 15%. When the first security has a return of 5%, the other has a return of 5% and holding both results in a return of 5%. The result of diversifying by holding both securities is a return of 15% half of the time and 5% the other half of the time, which is exactly the same set of returns that is earned by holding only one of the securities. Consequently, diversification in this case does not lead to any reduction of risk.

The examples we have just examined illustrate the following important points about diversification:

1. Diversification is almost always beneficial to the risk-averse investor since it reduces risk unless returns on securities move perfectly together (which is an extremely rare occurrence).
2. The less the returns on two securities move together, the more benefit (risk reduction) there is from diversification.

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[1]Such a case is described by saying that the returns on the two securities are perfectly *negatively* correlated.

[2]We can also see that diversification in the example above leads to lower risk by examining the standard deviation of returns when Irving diversifies and when he doesn't. The standard deviation of returns if Irving holds only one of the two securities is $\sqrt{0.5 \times (15\% - 10\%)^2 + 0.5 \times (5\% - 10\%)^2} = 5\%$. When Irving holds equal amounts of each security, there is a probability of $\frac{1}{4}$ that he will earn 5% on both (for a total return of 5%), a probability of $\frac{1}{4}$ that he will earn 15% on both (for a total return of 15%), and a probability of 1/2 that he will earn 15% on one and 5% on the other (for a total return of 10%). The standard deviation of returns when Irving diversifies is thus $\sqrt{0.25 \times (15\% - 10\%)^2 + 0.25 \times (5\% - 10\%)^2 + 0.5 \times (10\% - 10\%)^2} = 3.5\%$. Since the standard deviation of returns when Irving diversifies is lower than when he holds only one security, we can see that diversification has reduced risk.

Diversification and Beta

In the previous section, we demonstrated the benefits of diversification. Here, we examine diversification and the relationship between risk and returns in more detail. As a result, we obtain an understanding of two basic theories of asset pricing: the capital asset pricing model (CAPM) and arbitrage pricing theory (APT).

We start our analysis by considering a portfolio of n assets whose return is

$$R_p = x_1 R_1 1 + x_2 R_2 + \cdots + x_n R_n \tag{1}$$

where
R_p = the return on the portfolio of n assets
R_i = the return on asset i
x_i = the proportion of the portfolio held in asset i

The expected return on this portfolio, $E(R_p)$, equals

$$\begin{aligned} E(R_p) &= E(x_1 R_1) + E(x_2 R_2) + \cdots + E(x_n R_n) \\ &= x_1 E(R_1) + x_2 E(R_2) + \cdots + x_n E(R_n) \end{aligned} \tag{2}$$

An appropriate measure of the risk for this portfolio is the standard deviation of the portfolio's return (σ_p) or its squared value, the variance of the portfolio's return (σ_p^2), which can be written as

$$\begin{aligned} \sigma_p^2 &= E[R_p - E(R_p)]^2 = E[\{x_1 R_1 + \cdots + x_n R_n\} - \{x_1 E(R_1) + \cdots + x_n E(R_n)\}]^2 \\ &= E[x_1\{R_1 - E(R_1)\} + \cdots + x_n\{R_n - E(Rn)\}]^2 \end{aligned}$$

This expression can be rewritten as

$$\begin{aligned} \sigma_p^2 &= E[\{x_1[R_1 - E(R_1)] + \cdots + x_n[R_n - E(Rn)]\} \times \{R_p - E(R_p)\}] \\ &= x_1 E[\{R_1 - E(R_1)\} \times \{R_p - E(Rp)\}] + \cdots + x_n E[\{R_n - E(R_n)\} \times \{R_p - E(R_p)\}] \end{aligned}$$

This gives us the following expression for the variance for the portfolio's return:

$$\sigma_p^2 = x_1 \sigma_{1p} + x_2 \sigma_{2p} + x_n \sigma_{np} \tag{3}$$

where σ_{ip} = the covariance of the return on asset i with the portfolio's return $= E[\{R_i - E(R_i)\} \times \{R_p - E(Rp)\}]$

Equation 3 tells us that the contribution to risk of asset i to the portfolio is $x_i \sigma_{ip}$. By dividing this contribution to risk by the total portfolio risk (σ_p^2), we have the proportionate contribution of asset i to the portfolio risk:

$$x_i \sigma_{ip} / \sigma_p^2$$

The ratio σ_{ip}/σ_p^2 tells us about the sensitivity of asset i's return to the portfolio's return. The higher the ratio is, the more the value of the asset moves with changes in the value of the portfolio, and the more asset i contributes to portfolio risk. Our algebraic manipulations have thus led to the following important conclusion: ***The marginal contribution of an asset to the risk of a portfolio depends not***

on the risk of the asset in isolation, but rather on the sensitivity of that asset's return to changes in the value of the portfolio.

If the total of all risky assets in the market is included in the portfolio, then it is called the *market portfolio*. If we suppose that the portfolio, p, is the market portfolio, m, then the ratio σ_{im}/σ_m^2 is called the asset i's beta; that is:

$$\beta_i = \sigma_{im}/\sigma_m^2 \tag{4}$$

where β_i = the beta of asset i

An asset's beta, then, is a measure of the asset's marginal contribution to the risk of the market portfolio. A higher beta means that an asset's return is more sensitive to changes in the value of the market portfolio and that the asset contributes more to the risk of the portfolio.

Another way to understand beta is to recognize that the return on asset i can be considered as being made up of two components—one that moves with the market's return (R_m) and the other a random factor with an expected value of zero that is unique to the asset (ϵ_i) and so is uncorrelated with the market return:

$$R_i = \alpha_1 + \beta_i R_m + \epsilon_i \tag{5}$$

The expected return of asset i can then be written as

$$E(R_i) = \alpha_i + \beta_i E(R_m)$$

It is easy to show that β_i in the above expression is the beta of asset i we defined before by calculating the covariance of asset i's return with the market return using the two equations above:

$$\begin{aligned}\sigma_{im} &= E[\{R_i - E(R_i)\} \times \{R_m - E(R_m)\}] \\ &= E[\{\beta_i[R_m - E(R_m) + \epsilon_i\} \times \{R_m - E(R_m)\}]\end{aligned}$$

However, since ϵ_i is uncorrelated with R_m, $E[\{\epsilon_i\} \times \{R_m - E(R_m)\}] = 0$. Therefore,

$$\sigma_{im} = \beta_i \sigma_m^2$$

Dividing through by σ_m^2 gives us the following expression for β_i:

$$\beta_i = \sigma_{im}/\sigma_m^2$$

which is the same definition for beta we found in Equation 4.

The reason for demonstrating that the β_i in Equation 5 is the same as the one we defined before is that Equation 5 provides better intuition about how an asset's beta measures its sensitivity to changes in the market return. Equation 5 tells us that when the beta of an asset is 1.0, its return on average increases by 1 percentage point when the market return increases by 1 percentage point; when the beta is 2.0, the asset's return increases by 2 percentage points when the market return increases by 1 percentage point; and when the beta is 0.5, the asset's return only increases by 0.5 percentage point on average when the market return increases by 1 percentage point.

Equation 5 also tells us that we can get estimates of beta by comparing the average return on an asset with the average market return. For those of you who know a little econometrics, this estimate of beta is just an ordinary least squares regression of the asset's return on the market return. Indeed, the formula for the ordinary least squares estimate of $\beta_i = \sigma_{im}/\sigma_m^2$ is exactly the same as the definition of β_i earlier.

Systematic and Nonsystematic Risk

We can derive another important idea about the riskiness of an asset using Equation 5. The variance of asset i's return can be calculated from Equation 5 as

$$\sigma_i^2 = E[R_i - E(R_i)]^2 = E\{\beta_i[R_m - E(R_m)\} + \epsilon_i]^2$$

and since ϵ_i is uncorrelated with market return:

$$\sigma_i^2 = \beta_i^2\sigma_m^2 + \sigma_\epsilon^2$$

The total variance of the asset's return can thus be broken up into a component that is related to market risk, $\beta_i^2\sigma_m^2$, and a component that is unique to the asset, σ_ϵ^2. The $\beta_i^2\sigma_m^2$ component related to market risk is referred to as *systematic risk,* and the σ_ϵ^2 component unique to the asset is called *nonsystematic risk.* We can thus write the total risk of an asset as being made up of systematic risk and nonsystematic risk:

$$\text{Total asset risk} = \text{systematic risk} + \text{nonsystematic risk} \tag{6}$$

Systematic and nonsystematic risk each have another feature that makes the distinction between these two types of risk important. Systematic risk is the part of an asset's risk that cannot be eliminated by holding the asset as part of a diversified portfolio, whereas nonsystematic risk is the part of an asset's risk that can be eliminated in a diversified portfolio. Understanding these features of systematic and nonsystematic risk leads to the following important conclusion: ***The risk of a well-diversified portfolio depends only on the systematic risk of the assets in the portfolio.***

We can see that this conclusion is true by considering a portfolio of n assets, each of which has the same weight on the portfolio of $(1/n)$. Using Equation 5, the return on this portfolio is

$$R_p = (1/n)\sum_{i=1}^{n}\alpha_i + (1/n)\sum_{i=1}^{n}\beta_i R_m + (1/n)\sum_{i=1}^{n}\epsilon_i$$

which can be rewritten as

$$R_p = \overline{\alpha} + \overline{\beta}R_m + (1/n)\sum_{i=1}^{n}\epsilon_i$$

where $\quad \bar{\alpha} = \text{the average of the } \alpha_i\text{'s} = (1/n)\sum_{i=1}^{n}\alpha_i$

$$\bar{\beta} = \text{the average of the } \beta_i\text{'s} = (1/n)\sum_{i=1}^{n}\alpha_i$$

If the portfolio is well diversified so that the ϵ_i's are uncorrelated with each other, then using this fact and the fact that all the ϵ_i's are uncorrelated with the market return, the variance of the portfolio's return is calculated as

$$\sigma_p^2 = \bar{\beta}^2\sigma_m^2 + (1/n)(\text{average varience of } \epsilon_i)$$

As n gets large the second term, $(1/n)$(average variance of ϵ_i), becomes very small, so that a well-diversified portfolio has a risk of $\bar{\beta}^2\sigma_m^2$, which is only related to systematic risk. As the previous conclusion indicated, nonsystematic risk can be eliminated in a well-diversified portfolio. This reasoning also tells us that the risk of a well-diversified portfolio is greater than the risk of the market portfolio if the average beta of the assets in the portfolio is greater than 1; however, the portfolio's risk is less than the market portfolio if the average beta of the assets is less than 1.

The Capital Asset Pricing Model (CAPM)

We can now use the ideas we developed about systematic and nonsystematic risk and betas to derive one of the most widely used models of asset pricing—the capital asset pricing model (CAPM) developed by William Sharpe, John Litner, and Jack Treynor.

Each cross in Figure 1 shows the standard deviation and expected return for each risky asset. By putting different proportions of these assets into portfolios, we can generate a standard deviation and expected return for each of the portfolios using Equations 2 and 3. The shaded area in the figure shows these combinations of standard deviation and expected return for these portfolios. Since risk-averse investors always prefer to have higher expected return and lower standard deviation of the return, the most attractive standard deviation–expected return combinations are the ones that lie along the heavy line, which is called the *efficient portfolio frontier.* These are the standard deviation–expected return combinations risk-averse investors would always prefer.

The capital asset pricing model assumes that investors can borrow and lend as much as they want at a risk-free rate of interest, R_f. By lending at the risk-free rate, the investor earns an expected return of R_f and his investment has a zero standard deviation because it is risk-free. The standard deviation–expected return combination for this risk-free investment is marked as point A in Figure 1. Suppose an investor decides to put half of his total wealth in the risk-free loan and the other half in the portfolio on the efficient portfolio frontier with a standard deviation–expected return combination marked as point M in the figure. Using Equation 2, you should be able to verify that the expected return on this new portfolio is halfway between R_f and $E(R_m)$; that is, $[R_f + E(R_m)]/2$. Similarly, because the covariance between the risk-free return and the return on portfolio M must necessarily be zero, since there is no uncertainty about the return on the risk-free loan, you should also be able to verify, using Equation 3, that the standard deviation of the return on the new portfolio is halfway between zero and σ_m, that is, $(1/2)\sigma_m$.

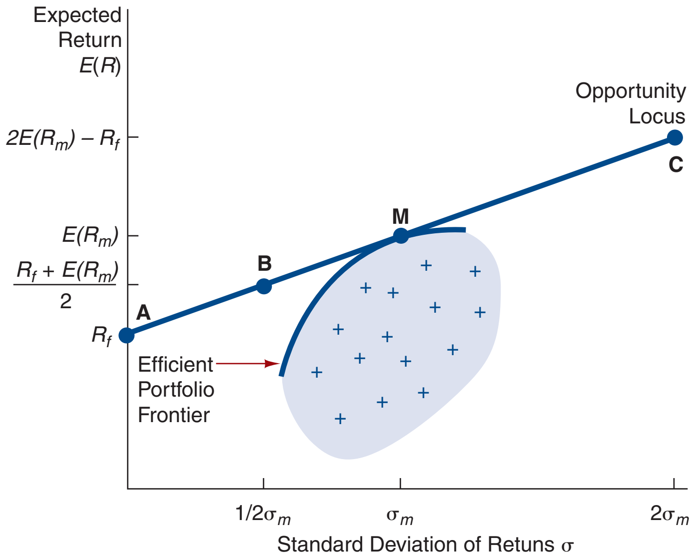

FIGURE 1 Risk Expected Return Trade-off

The crosses show the combination of standard deviation and expected return for each risky asset. The efficient portfolio frontier indicates the most preferable standard deviation –expected return combinations that can be achieved by putting risky assets into portfolios. By borrowing and lending at the risk-free rate and investing in portfolio M, the investor can obtain standard deviation–expected return combinations that lie along the line connecting A, B, M, and C. This line, the *opportunity locus,* contains the best combinations of standard deviations and expected returns available to the investor; hence the opportunity locus shows the trade-off between expected returns and risk for the investor.

The standard deviation–expected return combination for this new portfolio is marked as point B in the figure, and as you can see it lies on the line between point A and point M. Similarly, if an investor borrows the total amount of her wealth at the risk-free rate R_f and invests the proceeds plus her wealth (that is, twice her wealth) in portfolio M, then the standard deviation of this new portfolio will be twice the standard deviation of return on portfolio M, $2\sigma_m$. On the other hand, using Equation 2, the expected return on this new portfolio is $E(R_m) + E(R_m) - R_f$, which equals $2E(R_m) - R_f$. This standard deviation–expected return combination is plotted as point C in the figure.

You should now be able to see that both point B and point C are on the line connecting point A and point M. Indeed, by choosing different amounts of borrowing and lending, an investor can form a portfolio with a standard deviation–expected return combination that lies anywhere on the line connecting points A and M. You may have noticed that point M has been chosen so that the line connecting points A and M is tangent to the efficient portfolio frontier. The reason for choosing point M in this way is that it leads to standard deviation–expected return combinations along the line that are the most desirable for a risk-averse investor. This line can be thought of as the *opportunity locus,* which shows the best combinations of standard deviations and expected returns available to the investor.

The capital asset pricing model makes another assumption: All investors have the same assessment of the expected returns and standard deviations of all assets. In this case, portfolio M is the same for all investors. Thus when all investors' holdings of portfolio M are added together, they must equal all of the risky assets in the market, which is just the market portfolio. The assumption that all investors have the same assessment of risk and return for all assets thus means that portfolio M is the market portfolio.Therefore, the R_m and σ_m in Figure 1 are identical to the market return, R_m, and the standard deviation of this return, σ_m, referred to earlier in this appendix.

The conclusion that the market portfolio and portfolio M are one and the same means that the opportunity locus in Figure 1 can be thought of as showing the trade-off between expected returns and increased risk for the investor. This trade-off is given by the slope of the opportunity locus, $E(R_m) - R_f$, and it tells us that when an investor is willing to increase the risk of his portfolio by σ_m, then he can earn an additional expected return of $E(R_m) - R_f$. The market price of a unit of market risk, σ_m, is $E(R_m) - R_f$. $E(R_m) - R_f$ is therefore referred to as the *market price of risk.*

We now know that market price of risk is $E(R_m) - R_f$ and we also have learned that an asset's beta tells us about systematic risk, because it is the marginal contribution of that asset to a portfolio's risk. Therefore the amount an asset's expected return exceeds the risk-free rate, $E(R_i) - R_f$, should equal the market price of risk times the marginal contribution of that asset to portfolio risk, $[E(R_m) - R_f]\beta_i$. This reasoning yields the CAPM asset pricing relationship:

$$E(R_i) + \mathrm{R}_f + \beta_i[E(R_m) - R_f] \tag{7}$$

This CAPM asset pricing equation is represented by the upward sloping line in Figure 2, which is called the *security market line.* It tells us the expected return that the market sets for a security given its beta. For example, it tells us that if a security has a beta of 1.0 so that its marginal contribution to a portfolio's risk is the same as the market portfolio, then it should be priced to have the same expected return as the market portfolio, $E(R_m)$.

To see that securities should be priced so that their expected return–beta combination should lie on the security market line, consider a security like S in Figure 2, which is below the security market line. If an investor makes an investment in which half is put into the market portfolio and half into a risk-free loan, then the beta of this investment will be 0.5, the same as security S. However, this investment will have an expected return on the security market line that is greater than that for security S. Hence investors will not want to hold security S and its current price will fall, thus raising its expected return until it equals the amount indicated on the security market line. On the other hand, suppose there is a security like T which has a beta of 0.5 but whose expected return is above the security market line. By including this security in a well-diversified portfolio with other assets with a beta of 0.5, none of which can have an expected return less than that indicated by the security line (as we have shown), investors can obtain a portfolio with a higher expected return than that obtained by putting half into a risk-free loan and half into the market portfolio. This would mean that all investors would want to hold more of security T, and so its price would rise, thus lowering its expected return until it equaled the amount indicated on the security market line.

The capital asset pricing model formalizes the following important idea: ***An asset should be priced so that is has a higher expected return not when it has a greater risk in isolation, but rather when its systematic risk is greater.***

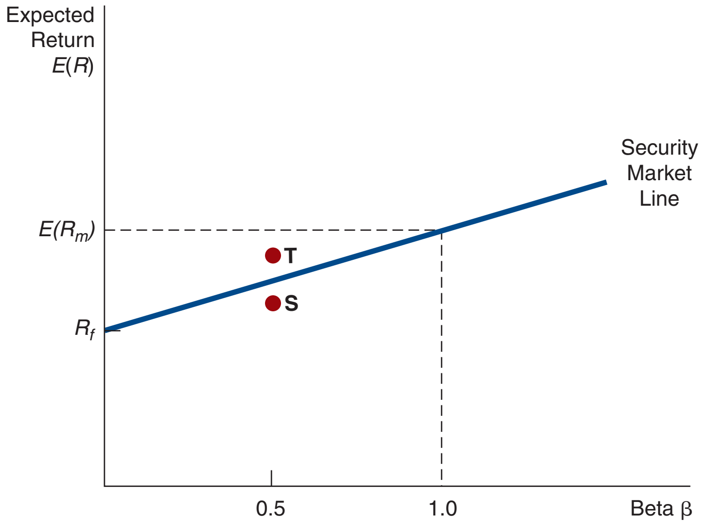

FIGURE 2 Security Market Line

The security market line derived from the capital asset pricing model describes the relationship between an asset's beta and its expected return.

Arbitrage Pricing Theory

Although the capital asset pricing model has proved to be very useful in practice, deriving it does require the adoption of some unrealistic assumptions; for example, the assumption that investors can borrow and lend freely at the risk-free rate, or the assumption that all investors have the same assessment of expected returns and standard deviations of returns for all assets. An important alternative to the capital asset pricing model is the arbitrage pricing theory (APT) developed by Stephen Ross of M.I.T.

In contrast to CAPM, which has only one source of systematic risk, the market return, APT takes the view that there can be several sources of systematic risk in the economy that cannot be eliminated through diversification. These sources of risk can be thought of as factors that may be related to such items as inflation, aggregate output, default risk premiums, and/or the term structure of interest rates. The return on an asset i can thus be written as being made up of components that move with these factors and a random component that is unique to the asset (ϵ_i):

$$R_i = \beta_i^1(\text{factor } 1) + \beta_i^2(\text{factor } 2) + \cdots + \beta_i^k(\text{factor } k) + \epsilon_i \tag{8}$$

Since there are k factors, this model is called a k-factor model. The $\beta_i^1, \cdots, \beta_i^k$ describe the sensitivity of the asset i's return to each of these factors.

Just as in the capital asset pricing model, these systematic sources of risk should be priced. The market price for each factor j can be thought of as $E(R_{\text{factor } j}) - R_f$, and hence the expected return on a security can be written as:

$$E(R_i) = R_f + \beta_i^1[E(R_{\text{factor } 1}) - R_f] + \cdots + \beta_i^k[E(R_{\text{factor } k}) - R_f] \tag{9}$$

This asset pricing equation indicates that all the securities should have the same market price for the risk contributed by each factor. If the expected return for a security were above the amount indicated by the APT pricing equation, then it would provide a higher expected return than a portfolio of other securities with the same average sensitivity to each factor. Hence investors would want to hold more of this security, and its price would rise until the expected return fell to the value indicated by the APT pricing equation. On the other hand, if the security's expected return were less than the amount indicated by the APT pricing equation, then no one would want to hold this security, because a higher expected return could be obtained with a portfolio of securities with the same average sensitivity to each factor. As a result, the price of the security would fall until its expected return rose to the value indicated by the APT equation.

As this brief outline of arbitrage pricing theory indicates, the theory supports a basic conclusion from the capital asset pricing model: An asset should be priced so that it has a higher expected return not when it has a greater risk in isolation, but rather when its systematic risk is greater. There is still substantial controversy about whether a variant of the capital asset pricing model or the arbitrage pricing theory is a better description of reality. At the present time, both frameworks are considered valuable tools for understanding how risk affects the prices of assets.