Math 70900 Midterm Exam Solutions

1. Prove that the set of all 2×2 matrices of rank one is a submanifold of the set of all 2×2 real matrices. What is its dimension?

   Solution: We identify 2×2 real matrices with $\mathbb{R}^4$ via $(w, x, y, z) \mapsto \left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right)$. Then such a matrix has rank two if and only if its determinant is nonzero, $wz - xy = 0$. It has rank zero if and only if it is the zero matrix. So the matrices where it has rank one are those with $wz - xy = 0$ and at least one of $\{w, x, y, z\}$ nonzero.

   The nonzero matrices form an open subset of $\mathbb{R}^4$ (deleting the origin), so we still have a manifold. The determinant function $\delta\colon \mathbb{R}^4 \backslash \{\mathbf{0}\} \to \mathbb{R}$ has every rank one matrix as a regular value, since
   $$D\delta = \begin{pmatrix} z & -y & -x & w \end{pmatrix},$$
   which has maximal rank one as long as at least one of $\{w, x, y, z\}$ is nonzero. Thus the set is a smooth submanifold using Theorem 9.1.2 (the Implicit Function Theorem).

2. The *Hopf fibration* is a map $F\colon S^3 \to S^2$ defined in the following way. We first define $\tilde{F}\colon \mathbb{R}^4 \to \mathbb{R}^3$ by
   $$\tilde{F}(w, x, y, z) = (w^2 + x^2 - y^2 - z^2, 2(wz - xy), 2(xz + wy))$$
   then restrict it to S^3.

   (a) Verify that if $\mathbf{x} \in S^3$, then $\tilde{F}(\mathbf{x}) \in S^2$, so that the restriction of $\tilde{F}$ actually is a smooth map from S^3 to S^2.

   Solution: A computation. Note that $(W - Z)^2 = (W + Z)^2 - 4WZ$ for any real numbers W, Z, so we can use $W = w^2 + x^2$ and $Z = y^2 + z^2$ to get a shortcut:
   $$\begin{aligned}
   &(w^2 + x^2 - y^2 - z^2)^2 + 4(wz - xy)^2 + 4(xz + wy)^2 \\
   &\qquad = (w^2 + x^2 + y^2 + z^2)^2 - 4(w^2 + x^2)(y^2 + z^2) \\
   &\qquad\qquad + 4(w^2z^2 - 2wxyz + x^2y^2) + 4(x^2z^2 + 2wxyz + w^2y^2) \\
   &\qquad = 1 - 4(w^2y^2 + w^2z^2 + x^2y^2 + x^2z^2) \\
   &\qquad\qquad + 4(w^2z^2 + x^2y^2 + x^2z^2 + w^2y^2) = 1.
   \end{aligned}$$

   So indeed it maps the sphere S^3 into the sphere S^2, and since those are both smooth submanifolds, the restriction of F is smooth.

   (b) Compute F_* on the left-invariant vector fields E_i from Homework #7, problem 3. Show that there are three vector fields V_i on S^2 that are F-related to E_i. (It is easier to do this for $\tilde{F}$ first.)

   Solution: The linear operator $\tilde{F}_* = D\tilde{F}$ is given on $\mathbb{R}^4$ by the matrix
   $$D\tilde{F} = \begin{pmatrix} 2w & 2x & -2y & -2z \\ 2z & -2y & -2x & 2w \\ 2y & 2z & 2w & 2x \end{pmatrix}.$$

Express coordinates on $\mathbb{R}^3$ as (t, u, v), so that

$$t = w^2 + x^2 - y^2 - z^2, \qquad u = 2(wz - xy), \qquad v = 2(wy + xz).$$

Using

$$\begin{aligned} E_1 &= -x \tfrac{\partial}{\partial w} + w \tfrac{\partial}{\partial x} + z \tfrac{\partial}{\partial y} - y \tfrac{\partial}{\partial z} \\ E_2 &= -y \tfrac{\partial}{\partial w} - z \tfrac{\partial}{\partial x} + w \tfrac{\partial}{\partial y} + x \tfrac{\partial}{\partial z} \\ E_3 &= -z \tfrac{\partial}{\partial w} + y \tfrac{\partial}{\partial x} - x \tfrac{\partial}{\partial y} + w \tfrac{\partial}{\partial z} \end{aligned}$$

as representatives in $\mathbb{R}^4$ as the three left-invariant basis vectors on S^3, we compute

$$D\tilde{F}(E_1) = \begin{pmatrix} 2w & 2x & -2y & -2z \\ 2z & -2y & -2x & 2w \\ 2y & 2z & 2w & 2x \end{pmatrix} \begin{pmatrix} -x \\ w \\ z \\ -y \end{pmatrix} = \begin{pmatrix} 0 \\ -4(wy + xz) \\ 4(wz - xy) \end{pmatrix} = \begin{pmatrix} 0 \\ -2v \\ 2u \end{pmatrix}$$

$$D\tilde{F}(E_2) = \begin{pmatrix} 2w & 2x & -2y & -2z \\ 2z & -2y & -2x & 2w \\ 2y & 2z & 2w & 2x \end{pmatrix} \begin{pmatrix} -y \\ -z \\ w \\ x \end{pmatrix} = \begin{pmatrix} -4(wy + xz) \\ 0 \\ 2(w^2 + x^2 - y^2 - z^2) \end{pmatrix} = \begin{pmatrix} -2v \\ 0 \\ 2t \end{pmatrix}$$

$$D\tilde{F}(E_3) = \begin{pmatrix} 2w & 2x & -2y & -2z \\ 2z & -2y & -2x & 2w \\ 2y & 2z & 2w & 2x \end{pmatrix} \begin{pmatrix} -z \\ y \\ -x \\ w \end{pmatrix} = \begin{pmatrix} -4(wz - xy) \\ 2(w^2 + x^2 - y^2 - z^2) \\ 0 \end{pmatrix} = \begin{pmatrix} -2u \\ 2t \\ 0 \end{pmatrix}.$$

We therefore see that

$$D\tilde{F}(E_1) = V_1, \qquad D\tilde{F}(E_2) = V_2, \qquad D\tilde{F}(E_3) = V_3,$$

where

$$V_1 = -2v \tfrac{\partial}{\partial u} + 2u \tfrac{\partial}{\partial v}, \qquad V_2 = -2v \tfrac{\partial}{\partial t} + 2t \tfrac{\partial}{\partial v}, \qquad V_3 = -2u \tfrac{\partial}{\partial t} + 2t \tfrac{\partial}{\partial u}.$$

These are all vector fields on $\mathbb{R}^3$ that are tangent to S^2, and we see that $F_*(E_i)\big|_p = (V_i)_{F(p)}$ for every $p \in S^3$, which is what it means to be F-related.

(c) Show that the rank of F is two everywhere, and describe the inverse images of points of S^2.

Solution: We need to know that $\{V_1, V_2, V_3\}$ always span a two-dimensional space. If $v \neq 0$, then V_1 and V_2 are linearly independent (since V_1 has a nonzero component in the $\frac{\partial}{\partial u}$ direction while V_2 is zero in that direction; and V_2 has a nonzero component in the $\frac{\partial}{\partial t}$ direction, while V_1 is zero in that direction). Similarly if $t \neq 0$ then V_2 and V_3 are linearly independent; and if $u \neq 0$ then V_1 and V_3 are linearly independent. Since $t^2 + u^2 + v^2 = 1$, at least one of these components is nonzero, so indeed F has maximal rank.

By the Implicit Function Theorem, since every point of S^2 is a regular point, the inverse images must be manifolds. They must be closed subsets of S^3 (hence compact), and must be one-dimensional manifolds, so they must all be circles.

3. Let $X = x\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y}$ and let $Y = (x+y)\,\frac{\partial}{\partial y}$ be vector fields on $\mathbb{R}^2$.

(a) Verify that $[X, Y] = 0$ everywhere by direct computation.
Solution: For any smooth function f we will have

$$\begin{aligned} X(Y(f)) &= x\partial_x((x+y)f_y) - x\partial_y((x+y)f_y) \\ &= x(x+y)f_{xy} + xf_y - x(x+y)f_{yy} - xf_y \\ &= x(x+y)(f_{xy} - f_{yy}) \\ Y(X(f)) &= (x+y)\partial_y(xf_x - xf_y) \\ &= (x+y)x(f_{xy} - f_{yy}), \end{aligned}$$

and these are the same. Hence $[X, Y](f) = 0$ on every function f.

(b) Find the flows of X and Y.
Solution: For the flow of X we need to solve the system

$$x'(t) = x(t), \qquad y'(t) = -x(t), \qquad x(0) = x_0, \quad y(0) = y_0.$$

The solution for the first equation is $x(t) = x_0 e^t$, and using this to get $y'(t) = -x_0 e^t$, we find that $y(t) = (x_0 + y_0) - x_0 e^t$. So the flow of X is

$$\Phi_t(x, y) = (xe^t, x + y - xe^t).$$

For the flow of Y we need to solve the system

$$x'(t) = 0, \qquad y'(t) = x(t) + y(t), \qquad x(0) = x_0, \quad y(0) = y_0.$$

Clearly $x(t) = x_0$, so that $y'(t) = x_0 + y(t)$, and so $y(t) = -x_0 + (x_0 + y_0)e^t$. Therefore the flow of Y is

$$\Psi_t(x, y) = (x, (x+y)e^t - x).$$

(c) Use the flows to construct coordinates (u, v) near the point $(1, 0)$ to make $X = \frac{\partial}{\partial u}$ and $Y = \frac{\partial}{\partial v}$.
Solution: The coordinates will be obtained indirectly by writing

$$(x, y) = \Phi_u\big(\Psi_v(1, 0)\big) = \Phi_u(1, e^v - 1) = (e^u, 1 + e^v - 1 - e^u) = (e^u, e^v - e^u).$$

Thus the transformation is $x = e^u$, $y = e^v - e^u$, or inverting $u = \ln x$, $v = \ln(y + x)$.
We verify by computing

$$\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} = e^u\frac{\partial}{\partial x} - e^u\frac{\partial}{\partial y} = x\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = X,$$

and similarly

$$\frac{\partial}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} = e^v\frac{\partial}{\partial y} = (x+y)\frac{\partial}{\partial y} = Y,$$

as desired.

4. (a) Given a point $p \in M$ and a vector $v \in T_pM$, show that there is a smooth vector field V on M with $V(p) = v$.

Solution: Choose a bump function ζ with support inside a coordinate chart containing p, which is constantly equal to 1 in a smaller neighborhood of p. This bump function can be smoothly extended to the entire manifold M by setting it to zero outside the coordinate chart.

Inside the coordinate chart, the vector v has an expression $v = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i}\big|_p$, and we define V by

$$V(q) = \begin{cases} \sum_{i=1}^n a^i \xi(q) \frac{\partial}{\partial x^i}\Big|_q & q \in \operatorname{supp}\zeta \\ 0_q & q \notin \operatorname{supp}\zeta. \end{cases}$$

Here 0_q denotes the zero vector in T_qM. Then V is a smooth vector field, and $V(p) = v$.

(b) Suppose $f\colon M \to \mathbb{R}$ is a smooth function with a critical point at p, i.e., such that $v(f) = 0$ for every $v \in T_pM$. If V is any smooth vector field and $w \in T_pM$ is any vector, show that the number $w(V(f))$ depends only on the vector $v = V(p)$, not on the choice of extension. Hence we get a bilinear operator $D^2f\colon T_pM \times T_pM \to \mathbb{R}$ given by $D^2f(w, v) = w(V(f))$ where V is any smooth extension of $v \in T_pM$.

Solution: Work in coordinates. THen $v(f) = 0$ for every $v \in T_pM$ means that $\frac{\partial}{\partial x^i}\big|_p = 0$, so all the first derivatives of f are zero at p. Now if V is a general vector field defined in a neighborhood of p, we can write

$$V_q = \sum_{i=1}^n a^i(q) \frac{\partial}{\partial x^i}\Big|_q,$$

so that the function $V(f)$ in a neighborhood of p is given by

$$V(f)(q) = \sum_{i=1}^n a^i(q) \frac{\partial f}{\partial x^i}\Big|_q,$$

for some functions a^i. Now write

$$w = \sum_{j=1}^n b^j \frac{\partial}{\partial x^j}\Big|_p,$$

and we compute that

$$\begin{aligned} w(V(f)) &= \sum_{i=1}^n \sum_{j=1}^n b^j \frac{\partial}{\partial x^j}\Big(a^i(q) \frac{\partial f}{\partial x^i}(q)\Big)\Big|_{q=p} \\ &= \sum_{i,j=1}^n b^j \frac{\partial a^i}{\partial x^j}\Big|_p \frac{\partial f}{\partial x^i}\Big|_p + b^j a^i(p) \frac{\partial^2 f}{\partial x^i \partial x^j}\Big|_p. \end{aligned}$$

Since $\frac{\partial f}{\partial x^i}\big|_p = 0$ since f has a critical point at p, the first term disappears, and the second term only depends on b^j and $a^i(p)$.

(c) Show that $D^2 f$ is symmetric.

Solution: A simple way is to use the coordinate expression derived in the previous part to see this bilinear operator's coefficients are $\frac{\partial^2 f}{\partial x^i \partial x^j}$, which is symmetric between i and j.

For a more sophisticated proof, extend v to a vector field V, and extend w to a vector field W. Then $D^2 f(v,w) = W(V(f))|_p$ while $D^2 f(w,v) = V(W(f))|_p$. The difference between these is

$$W(V(f))|_p - V(W(f))|_p = [W,V](f)|_p = 0,$$

since $[W,V]$ is a vector field and thus a first-order differential operator; thus it vanishes on f at p because it has a critical point.

(d) Compute $D^2 f$ at the origin if $f\colon \mathbb{C} \to \mathbb{R}$ is given by $f(z) = \mathrm{Re}(z^2)$.

Solution: In coordinates we have $f(x,y) = x^2 - y^2$, so that the first derivatives are zero at the origin, and the second derivatives are $\frac{\partial^2 f}{\partial x^2} = 2$, $\frac{\partial^2 f}{\partial x \partial y} = 0$, $\frac{\partial^2 f}{\partial y^2} = -2$. We obtain the bilinear operator

$$D^2 f = 2\, dx \otimes dx - 2\, dy \otimes dy.$$

In other words, $D^2 f$ is determined by its action on the basis by

$$D^2 f(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x}) = 2, \qquad D^2 f(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}) = 0, \qquad D^2 f(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial y}) = -2.$$

5. Show that the tangent bundle TM is always orientable as a manifold even if M itself is not.

Solution: Our definition of orientable is in terms of coordinate transition map determinants, so we need to compute those.

Recall that coordinates on TM were constructed by starting with a coordinate chart (ϕ, U) on M given by $(x^1, \ldots, x^n)$ and writing an arbitrary vector $v \in T_pM \subset TU$ uniquely as

$$v = \sum_{i=1}^n a^i \, \frac{\partial}{\partial x^i}\Big|_p.$$

The coordinates of v are then given by

$$(\phi(p), a^1, \ldots, a^n) = (x^1, \ldots, x^n, a^1, \ldots, a^n).$$

If (ψ, V) is another coordinate chart given by $(y^1, \ldots, y^n)$, then the vector v can be written as

$$v = \sum_{i=1}^n \sum_{j=1}^n a^i \, \frac{\partial y^j}{\partial x^i} \, \frac{\partial}{\partial y^j}\Big|_p = \sum_{j=1}^n b^j \, \frac{\partial}{\partial y^j}\Big|_p,$$

so that the transition map to express (y, b) in terms of (x, a) is given by

$$(y^1, \ldots, y^n, b^1, \ldots, b^n) = \left(\psi \circ \phi^{-1}(x^1, \ldots, x^n), \sum_{k=1}^n a^k \, \frac{\partial y^1}{\partial x^k}, \ldots, \sum_{k=1}^n a^k \, \frac{\partial y^n}{\partial x^k} \right).$$

We compute what the Jacobian determinant of this is, which involves first computing the $2n$ possible partial derivatives (with respect to all x^i and then with respect to all a^i). Note that the first half of the transformation depends only on the x^i, not on the a^i, and so we get a block matrix form with the upper right block (the derivatives of y^j with respect to a^i) all zeroes. Meanwhile the upper left block is the usual Jacobian matrix $J_i^j := \frac{\partial y^j}{\partial x^i}$.

For the bottom right corner, we differentiate the b^j with respect to the a^i, and since these equations are *linear*, the only term we get is

$$\frac{\partial b^j}{\partial a^i} = \frac{\partial}{\partial a^i} \sum_{k=1}^n a^k \frac{\partial y^j}{\partial x^k} = \frac{\partial y^j}{\partial x^i} = J_i^j.$$

That is, we see exactly the same Jacobian matrix in the bottom right corner as in the upper left corner.

Finally we can compute the bottom left corner, which differentiates the b^j with respect to x^i. This is a complicated expression which we will denote by

$$K_i^j = \sum_{k=1}^n a^k \frac{\partial^2 y^j}{\partial x^i \partial x^k}.$$

The point though is that K doesn't matter, since the full Jacobian transformation matrix is of the form

$$\mathbf{J} = \begin{pmatrix} J & 0 \\ K & J \end{pmatrix}.$$

The determinant of this is given by $\det \mathbf{J} = (\det J)^2$, which is always positive regardless of whether $\det J$ is positive or negative. This proves that our standard coordinate charts on TM are automatically compatible with each other.

6. Consider the group $\mathbb{R}^3$ with operation

$$(x, y, z) \cdot (u, v, w) = (x + e^z u, y + e^{-z} v, z + w).$$

Compute a basis of left-invariant vector fields $\{E_1, E_2, E_3\}$ in (x, y, z) coordinates, and determine the Lie brackets $[E_i, E_j] = \sum_{k=1}^3 c_{ijk} E_k$.

Solution: First we compute the left translation map and its differential (planning to rename all coordinates later). Fix (x, y, z) and denote (a, b, c) as a point in the domain of $L_{(x,y,z)}$ and (u, v, w) as a point in the range of $L_{(x,y,z)}$. Then we get

$$(u, v, w) = (x, y, z) \cdot (a, b, c) = (x + e^z a, y + e^{-z} b, z + c).$$

The push-forward by $L_{(x,y,z)}$ is then given by differentiating (u, v, w) with respect to (a, b, c), to get

$$\begin{aligned}
(L_{(x,y,z)})_* \left(\tfrac{\partial}{\partial a}\big|_{(a,b,c)} \right) &= \tfrac{\partial u}{\partial a} \tfrac{\partial}{\partial u}\big|_{(u,v,w)} + \tfrac{\partial v}{\partial a} \tfrac{\partial}{\partial v}\big|_{(u,v,w)} + \tfrac{\partial w}{\partial a} \tfrac{\partial}{\partial w}\big|_{(u,v,w)} = e^z \tfrac{\partial}{\partial u}\big|_{(u,v,w)} \\
(L_{(x,y,z)})_* \left(\tfrac{\partial}{\partial b}\big|_{(a,b,c)} \right) &= \tfrac{\partial u}{\partial b} \tfrac{\partial}{\partial u}\big|_{(u,v,w)} + \tfrac{\partial v}{\partial b} \tfrac{\partial}{\partial v}\big|_{(u,v,w)} + \tfrac{\partial w}{\partial b} \tfrac{\partial}{\partial w}\big|_{(u,v,w)} = e^{-z} \tfrac{\partial}{\partial v}\big|_{(u,v,w)} \\
(L_{(x,y,z)})_* \left(\tfrac{\partial}{\partial c}\big|_{(a,b,c)} \right) &= \tfrac{\partial u}{\partial c} \tfrac{\partial}{\partial u}\big|_{(u,v,w)} + \tfrac{\partial v}{\partial c} \tfrac{\partial}{\partial v}\big|_{(u,v,w)} + \tfrac{\partial w}{\partial c} \tfrac{\partial}{\partial w}\big|_{(u,v,w)} = \tfrac{\partial}{\partial w}\big|_{(u,v,w)}.
\end{aligned}$$

Now to get the left-invariant fields, we suppose that (a, b, c) is the identity element $(0, 0, 0)$, so that $(u, v, w) = (x, y, z)$, and get

$$\begin{aligned}
E_1 &= (L_{(x,y,z)})_*\big(\tfrac{\partial}{\partial x}\big|_{(0,0,0)}\big) = e^z \,\tfrac{\partial}{\partial x}\big|_{(x,y,z)} \\
E_2 &= (L_{(x,y,z)})_*\big(\tfrac{\partial}{\partial y}\big|_{(0,0,0)}\big) = e^{-z} \,\tfrac{\partial}{\partial y}\big|_{(x,y,z)} \\
E_3 &= (L_{(x,y,z)})_*\big(\tfrac{\partial}{\partial z}\big|_{(0,0,0)}\big) = \tfrac{\partial}{\partial z}\big|_{(x,y,z)}.
\end{aligned}$$

To compute the Lie brackets, we use the commutators and apply them to functions. Abbreviating $E_1 = e^z\,\partial_x$, $E_2 = e^{-z}\,\partial_y$, and $E_3 = \partial_z$, we get

$$\begin{aligned}
&[E_1, E_2](f) = e^z\partial_x(e^{-z} f_y) - e^{-z}\partial_y(e^z f_x) = f_{yx} - f_{xy} = 0 \\
&[E_2, E_3](f) = e^{-z}\partial_y(f_z) - \partial_z(e^{-z} f_y) = e^{-z} f_{zy} + e^{-z} f_y - e^{-z} f_{yz} = e^{-z} f_y = E_2(f) \\
&[E_3, E_1](f) = \partial_z(e^z f_x) - e^z\partial_x(f_z) = e^z f_x + e^z f_{xz} - e^z f_{zx} = e^z f_x = E_1(f).
\end{aligned}$$

We conclude that the Lie algebra is determined by

$$[E_1, E_2] = 0, \qquad [E_2, E_3] = E_2, \qquad [E_3, E_1] = E_1.$$