Math 70900 Homework #9 Solutions

1. Let X be a vector field on a two-dimensional manifold M and let $\gamma\colon(-\varepsilon,\varepsilon)\to M$ be a curve such that $\gamma'(s)$ and $X_{\gamma(s)}$ are always linearly independent. (In the language of partial differential equations one says that γ is *non-characteristic*.) Show that given any function $h\colon(-\varepsilon,\varepsilon)\to\mathbb{R}$, there is a neighborhood U of $p=\gamma(0)$ and a smooth function $f\colon U\to\mathbb{R}$ such that $X(f)=0$ on U and $f(\gamma(s))=h(s)$ for $s\in(-\varepsilon,\varepsilon)$. This is generically how one solves a first-order PDE.

   Solution: Choose coordinates (s,t) on M near p such that $X=\frac{\partial}{\partial t}$ and s is the parameter along the curve γ. That is, we map $(s,t)\mapsto\Phi_t(\gamma(s))$ for (s,t) sufficiently small. Since $\frac{\partial}{\partial s}$ and $X=\frac{\partial}{\partial t}$ are linearly independent at $(0,0)$ by assumption (in fact for all $s\in(-\varepsilon,\varepsilon)$ when $t=0$), we know this will be a smooth and locally invertible map into some other coordinates, which can therefore serve as a coordinate system locally.

   Since $X=\frac{\partial}{\partial t}$, we know a function satisfying $X(f)=0$ on an open set must be only a function of s. So we extend f on the whole (s,t) coordinate chart by using the same formula $f\big(\Phi_t(\gamma(s))\big)=h(s)$, i.e., having f depend only on s. This function is independent of t so it satisfies $X(f)=0$, and when $t=0$ it has the correct values on the initial curve γ.

2. For the 1-form ω on $\mathbb{R}^4$ given by $\omega=xy\,dw-wz\,dx+y^2\,dy-zx\,dz$, compute $d\omega$.

   Solution: Using the product rule we have

$$\begin{aligned}d\omega&=d(xy)\wedge dw-d(wz)\wedge dx+d(y^2)\wedge dy-d(zx)\wedge dz\\&=x\,dy\wedge dw+y\,dx\wedge dw-w\,dz\wedge dx-z\,dw\wedge dx+2y\,dy\wedge dy-z\,dx\wedge dz-x\,dz\wedge dz\\&=x\,dy\wedge dw-(y+z)\,dw\wedge dx+(w-z)\,dx\wedge dz,\end{aligned}$$

   using antisymmetric of wedge products of 1-forms.

3. Imitate the proof of Proposition 15.2.10 to show that if ω is a 1-form on $\mathbb{R}^3$ with $d\omega=0$, then $\omega=df$ for some smooth function $f\colon\mathbb{R}^3\to\mathbb{R}$.

   Solution: We just need to come up with a formula that works for f. Write

$$\omega(x,y,z)=p(x,y,z)\,dx+q(x,y,z)\,dy+r(x,y,z)\,dz,$$

   where we assume $d\omega=0$, so that

$$\frac{\partial p}{\partial z}=\frac{\partial r}{\partial x},\qquad\frac{\partial q}{\partial x}=\frac{\partial p}{\partial y},\qquad\frac{\partial r}{\partial y}=\frac{\partial q}{\partial z}.$$

   Inspired by the formula in that Proposition, we try

$$f(x,y,z)=\int_0^x p(s,y,z)\,ds+\int_0^y q(0,s,z)\,ds+\int_0^z r(0,0,s)\,ds.$$

Differentiating with respect to x, we obviously get

$$\frac{\partial f}{\partial x}(x, y, z) = p(x, y, z).$$

Differentiating with respect to y and using the fact that $d\omega = 0$, we get

$$\begin{aligned}
\frac{\partial f}{\partial y}(x, y, z) &= \int_0^x \frac{\partial p}{\partial y}(s, y, z)\, ds + q(0, y, z) \\
&= \int_0^x \frac{\partial q}{\partial s}(s, y, z)\, ds + q(0, y, z) \\
&= q(x, y, z) - q(0, y, z) + q(0, y, z) = q(x, y, z).
\end{aligned}$$

Similarly for the z-derivative, we use again the fact that $d\omega = 0$ to get

$$\begin{aligned}
\frac{\partial f}{\partial z}(x, y, z) &= \int_0^x \frac{\partial p}{\partial z}(s, y, z)\, ds + \int_0^y \frac{\partial q}{\partial z}(0, s, z)\, ds + r(0, 0, z) \\
&= \int_0^x \frac{\partial r}{\partial s}(s, y, z)\, ds + \int_0^y \frac{\partial r}{\partial s}(0, s, z)\, ds + r(0, 0, z) \\
&= r(x, y, z) - r(0, y, z) + r(0, y, z) - r(0, 0, z) = r(0, 0, z) \\
&= r(x, y, z).
\end{aligned}$$

We have therefore found a function $f(x, y, z)$ with $df = \omega$ everywhere.

4. A *contact form* on a 3-dimensional manifold is a 1-form α such that $\alpha \wedge d\alpha$ is never zero.

   (a) Show that $\alpha = dz - x\, dy$ is a contact form on $\mathbb{R}^3$.
   Solution: We have $d\alpha = -dx \wedge dy$, so that

   $$\alpha \wedge d\alpha = (dz - x\, dy) \wedge (-dx) \wedge (dy) = -dz \wedge dx \wedge dy = -dx \wedge dy \wedge dz.$$

   (b) Show that $\alpha = \sin z\, dx + \cos z\, dy$ is a contact form on $\mathbb{R}^3$ which descends to a contact form on $\mathbb{T}^3$.
   Solution: We have

   $$d\alpha = \cos z\, dz \wedge dx + \sin z\, dy \wedge dz.$$

   Wedging α with this, we get

   $$\alpha \wedge d\alpha = \cos^2 z\, dy \wedge dz \wedge dx + \sin^2 z\, dx \wedge dy \wedge dz = dx \wedge dy \wedge dz,$$

   which is never zero.
   The 1-form α is invariant under the group action $(x, y, z) \mapsto (x + 2j\pi, y + 2k\pi, z + 2n\pi)$ for integers j, k, n, and thus there is a 1-form ω on $\mathbb{T}^3$ such that if $P\colon \mathbb{R}^3 \to \mathbb{T}^3$ is the quotient projection, we have $P^{\#}\omega = \alpha$.

(c) Show that if α is a contact form, then there is a unique vector field ξ (called the *Reeb field*) such that $d\alpha(\xi, u) = 0$ for every vector u, and $\alpha(\xi) = 1$ everywhere.

Solution: Express $\alpha = p\,dx + q\,dy + r\,dz$ in coordinates, so that

$$d\alpha = (r_y - q_z)\,dy \wedge dz + (p_z - r_x)\,dz \wedge dx + (q_x - p_y)\,dx \wedge dy.$$

Then the wedge product is

$$\alpha \wedge d\alpha = \big[p(r_y - q_z) + q(p_z - r_x) + r(q_x - p_y)\big]\,dx \wedge dy \wedge dz,$$

and by assumption this coefficient is nonzero.

Write the Reeb field as $\xi = f\,\partial_x + g\,\partial_y + h\,\partial_z$. Then the condition $d\alpha(\xi, u) = 0$ for every vector field u is equivalent to the three equations (choosing $u = \partial_x$, $u = \partial_y$, and $u = \partial_z$ successively)

$$\begin{aligned} (p_z - r_x)h - (q_x - p_y)g &= 0 \\ -(r_y - q_z)h + (q_x - p_y)f &= 0 \\ (r_y - q_z)g - (p_z - r_x)f &= 0. \end{aligned}$$

Consider this as a linear system for $\{f, g, h\}$. If all the components of this system were zero at some point, then $d\alpha$ would be zero at that point, so $\alpha \wedge d\alpha \neq 0$ would be impossible. Thus at least one is nonzero, and we may assume (by permuting the variables if needed) that it's $p_z - r_x \neq 0$. Then we have

$$h = \frac{q_x - p_y}{p_z - r_x}\,g, \qquad f = \frac{r_y - q_z}{p_z - r_x}\,g.$$

Hence in this case g determines the other components, and we see that any Reeb vector ξ must be a multiple of the vector

$$\zeta = (r_y - q_z)\,\partial_x + (p_z - r_x)\,\partial_y + (q_x - p_y)\,\partial_z.$$

So we must have $\xi = \varphi\zeta$ for some function φ, which is determined by the extra condition $\alpha(\xi) = 1$, which is equivalent to

$$\varphi\big[p(r_y - q_z) + q(p_z - r_x) + r(q_x - p_y)\big] = 1.$$

By assumption the term in square brackets is everywhere nonzero, and so φ is a uniquely determined smooth function in these coordinates.

Since we have a uniquely specified formula for the Reeb field in any set of coordinates, which are written in terms of the smooth components of α, we see that ξ is uniquely determined and smooth in any chart, and therefore globally on the manifold M.

(d) Find the Reeb field for the contact form in part (b).

Solution: We have already essentially worked out the formula in general. Here we have $p = \sin z$ and $q = \cos z$, with $r = 0$, so the field ζ is given by

$$\zeta = (r_y - q_z)\,\partial_x + (p_z - r_x)\,\partial_y + (q_x - p_y)\,\partial_z = \sin z\,\partial_x + \cos z\,\partial_y.$$

Since we already see that

$$\alpha(\zeta) = \sin^2 z + \cos^2 z = 1,$$

we see that $\varphi \equiv 1$ and ζ is already the Reeb field ξ.

5. Let $\eta\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the diffeomorphism $\eta(u,v) = (2v - u^2, 3u, 4u + v^2)$, and let $\omega = y\,dx \wedge dy - z\,dz \wedge dx + x\,dy \wedge dz$. Compute $\eta^\# \omega$.

   Solution: There are two ways to go: either from the definition (by seeing what $\eta^\#$ does to pairs of vector fields) or using the shortcut formulas: including the fact that d commutes with $\eta^\#$ and so does the wedge product. The latter is almost always easier.

   We get $x \circ \eta(u,v) = 2v - u^2$, $y \circ \eta(u,v) = 3u$, and $z \circ \eta(u,v) = 4u + v^2$. Therefore we find

$$\begin{aligned} \eta^\# dx &= -2u\,du + 2\,dv \\ \eta^\# dy &= 3\,du \\ \eta^\# dz &= 4\,du + 2v\,dv. \end{aligned}$$

   Thus we get

$$\begin{aligned} \eta^\#(dx \wedge dy) &= (-2u\,du + 2\,dv) \wedge (3\,du) = -6\,du \wedge dv \\ \eta^\#(dz \wedge dx) &= (4\,du + 2v\,dv) \wedge (-2u\,du + 2\,dv) = (8 + 4uv)\,du \wedge dv \\ \eta^\#(dy \wedge dz) &= 6v\,du \wedge dv. \end{aligned}$$

   Combining, we obtain

$$\begin{aligned} \eta^\# \omega &= (y \circ \eta)\eta^\#(dx \wedge dy) - (z \circ \eta)\eta^\#(dz \wedge dx) + (x \circ \eta)\eta^\#(dy \wedge dz) \\ &= 3u(-6)\,du \wedge dv - (4u + v^2)(8 + 4uv)\,du \wedge dv + (2v - u^2)(6v)\,du \wedge dv \\ &= (-18u - 32u - 8v^2 - 16u^2 v - 4uv^3 + 12v^2 - 6u^2 v)\,du \wedge dv \\ &= (-50u + 4v^2 - 22u^2 v - 4uv^3)\,du \wedge dv. \end{aligned}$$

6. Let $\gamma\colon [0,1] \to \mathbb{R}^3$ be $\gamma(t) = \big(t, t^2, t^3\big)$, and let $\omega = z\,dx - x\,dy + y\,dz$. Compute $\int_\gamma \omega$.

   Solution: By definition we have

$$\int_\gamma \omega = \int_0^1 \gamma^\# \omega.$$

   We compute that $\gamma^\# dx = dt$, $\gamma^\# dy = 2t\,dt$, and $\gamma^\# dz = 3t^2\,dt$. Therefore we get

$$\gamma^\# \omega = t^3\,dt - t(2t)\,dt + t^2(3t^2)\,dt = (3t^4 + t^3 - 2t^2)\,dt.$$

   Integrating this over $[0,1]$, we get

$$\int_\gamma \omega = \tfrac{3}{5} + \tfrac{1}{4} - \tfrac{2}{3} = \tfrac{11}{60}.$$