

Read “Introduction to Differential Geometry” through Chapter 17, Section

1.

1. Let  $X$  be a vector field on a two-dimensional manifold  $M$  and let  $\gamma: (-\varepsilon, \varepsilon) \rightarrow M$  be a curve such that  $\gamma'(s)$  and  $X_{\gamma(s)}$  are always linearly independent. (In the language of partial differential equations one says that  $\gamma$  is *non-characteristic*.) Show that given any function  $h: (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ , there is a neighborhood  $U$  of  $p = \gamma(0)$  and a smooth function  $f: U \rightarrow \mathbb{R}$  such that  $X(f) = 0$  on  $U$  and  $f(\gamma(s)) = h(s)$  for  $s \in (-\varepsilon, \varepsilon)$ . This is generically how one solves a first-order PDE.
2. For the 1-form  $\omega$  on  $\mathbb{R}^4$  given by  $\omega = xy dw - wz dx + y^2 dy - zx dz$ , compute  $d\omega$ .
3. Imitate the proof of Proposition 15.2.10 to show that if  $\omega$  is a 1-form on  $\mathbb{R}^3$  with  $d\omega = 0$ , then  $\omega = df$  for some smooth function  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ .
4. A *contact form* on a 3-dimensional manifold is a 1-form  $\alpha$  such that  $\alpha \wedge d\alpha$  is never zero.
  - (a) Show that  $\alpha = dz - x dy$  is a contact form on  $\mathbb{R}^3$ .
  - (b) Show that  $\alpha = \sin z dx + \cos z dy$  is a contact form on  $\mathbb{R}^3$  which descends to a contact form on  $\mathbb{T}^3$ .
  - (c) Show that if  $\alpha$  is a contact form, then there is a unique vector field  $\xi$  (called the *Reeb field*) such that  $d\alpha(\xi, u) = 0$  for every vector  $u$ , and  $\alpha(\xi) = 1$  everywhere.
  - (d) Find the Reeb field for the contact form in part (b).
5. Let  $\eta: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be the diffeomorphism  $\eta(u, v) = (2v - u^2, 3u, 4u + v^2)$ , and let  $\omega = y dx \wedge dy - z dz \wedge dx + x dy \wedge dz$ . Compute  $\eta^\# \omega$ .
6. Let  $\gamma: [0, 1] \rightarrow \mathbb{R}^3$  be  $\gamma(t) = (t, t^2, t^3)$ , and let  $\omega = z dx - x dy + y dz$ . Compute  $\int_\gamma \omega$ .