Math 70900 Homework #8 Solutions

1. On $M = \mathbb{R}^2$, let $X = y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y}$ and $Y = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}$.

   (a) Show that $[X, Y] = 0$.
   Solution: For any function f, we have

   $$X(Y(f)) = y\partial_x(xf_x+yf_y)+x\partial_y(xf_x+yf_y) = yf_x+xyf_{xx}+y^2f_{xy}+x^2f_{xy}+xf_y+xyf_{yy},$$

   while

   $$Y(X(f)) = x\partial_x(yf_x+xf_y)+y\partial_y(yf_x+xf_y) = xyf_{xx}+xf_y+x^2f_{xy}+yf_x+y^2f_{xy}+xyf_{yy}.$$

   Subtracting gives

   $$[X, Y](f) = yf_x + xf_y - xf_y - yf_x = 0.$$

   Since the bracket is zero on any function, the vector field $[X, Y]$ is zero everywhere.

   (b) Find the flows Φ_t of X and Ψ_t of Y.
   Solution: The flow of X is found from solving the system

   $$x'(t) = y(t), \qquad y'(t) = x(t), \qquad x(0) = x_0, \quad y(0) = y_0.$$

   This reduces to $x''t) = x(t)$, with general solution $x(t) = A\cosh t + B\sinh t$, so that $y(t) = x'(t) = A\sinh t + B\cosh t$. Thus we must have $A = x_0$ and $B = y_0$, so the flow is

   $$\Phi_t(x, y) = (x\cosh t + y\sinh t, x\sinh t + y\cosh t).$$

   The flow of Y is found from solving

   $$x'(t) = x(t), \qquad y'(t) = y(t), \qquad x(0) = x_0, \quad y(0) = y_0,$$

   with solution $x(t) = x_0e^t$, $y(t) = y_0e^t$. Thus we have

   $$\Psi_t(x, y) = (xe^t, ye^t).$$

   (c) Construct an explicit coordinate chart near $(1, 0)$ such that $X = \frac{\partial}{\partial u}$ and $Y = \frac{\partial}{\partial v}$.
   Solution: Starting from the point $(1, 0)$, we define a map from (u, v) to (x, y) by

   $$(x, y) = \Phi_u\big(\Psi_v(1, 0)\big) = \Phi_u(e^v, 0) = (e^v\cosh u, e^v\sinh u).$$

   We check that

   $$\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} = e^v\sinh u\,\frac{\partial}{\partial x} + e^v\cosh u\,\frac{\partial}{\partial y} = y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y} = X,$$

   and similarly

   $$\frac{\partial}{\partial v} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} = e^v\cosh u\,\frac{\partial}{\partial x} + e^v\sinh u\,\frac{\partial}{\partial y} = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} = Y,$$

2. Suppose X and Y are vector fields on M, with flows Φ_t and Ψ_t respectively. Prove that
$$[X,Y]_p = -\frac{\partial}{\partial t}\Big|_{t=0}\frac{\partial}{\partial s}\Big|_{s=0}\Phi_t\circ\Psi_s\circ\Phi_{-t}\circ\Psi_{-s}(p).$$

Solution: By the Chain Rule, we have
$$\frac{\partial}{\partial s}\Phi_t\circ\Psi_s\circ\Phi_{-t}\circ\Psi_{-s}(p) = (\Phi_t)_*\left(\frac{\partial}{\partial s}\Psi_s\circ\Phi_{-t}\circ\Psi_{-s}(p)\right). \tag{1}$$
Fix p and write $\gamma(s)=\Psi_{-s}(p)$ and $F=\Phi_{-t}$; then we need to compute
$$\frac{\partial}{\partial s}\Psi\big(s, F(\gamma(s))\big),$$
and using the Chain Rule again, we have
$$\begin{aligned}\frac{\partial}{\partial s}\Psi\big(s,F(\gamma(s))\big) &= \frac{\partial\Psi}{\partial s}\big(s,F(\gamma(s))\big) + (\Psi_s)_*\left(\frac{\partial}{\partial s}F(\gamma(s))\right)\\ &= Y_{\Psi_s(F(\gamma(s)))} + (\Psi_s)_*\Big(F_*\big(\gamma'(s)\big)\Big).\end{aligned}$$

This is a bit complicated, but plugging in $s=0$ makes it easier: we get $\gamma(0)=\Psi_{-s}(p)=p$ and
$$\gamma'(0) = \frac{\partial}{\partial s}\Psi_{-s}(p)\Big|_{s=0} = -Y_{\Psi_{-s}(p)}\Big|_{s=0} = -Y_p,$$
since Ψ_0 is the identity map. Therefore
$$\frac{\partial}{\partial s}\Big|_{s=0}\Psi\big(s,F(\gamma(s))\big)Y_{F(p)} + F_*\big(-Y_p\big) = Y_{\Phi_{-t}(p)} - (\Phi_{-t})_*(Y_p).$$
Plugging into equation (1) gives
$$\begin{aligned}\frac{\partial}{\partial s}\Big|_{s=0}\Phi_t\circ\Psi_s\circ\Phi_{-t}\circ\Psi_{-s}(p) &= (\Phi_t)_*\left(Y_{\Phi_{-t}(p)} - (\Phi_{-t})_*(Y_p)\right)\\ &= (\Phi_t)_*(Y_{\Phi_{-t}(p)}) - Y_p = ((\Phi_t)_\# Y)_p - Y_p.\end{aligned}$$

Now take the time derivative of this with respect to t (in each tangent space T_pM) to get
$$\frac{\partial}{\partial t}\Big|_{t=0}(\Phi_t)_\# Y - Y = -\mathcal{L}_X Y$$
by the definition in Proposition 14.5.4, which is equal to $-[X,Y]$ also by Proposition 14.5.4.

An alternate way to do this is expanding locally in s and t as we did in class up to first order.

3. Let $G=GL_n(\mathbb{R})$ be the Lie group of all real invertible $n\times n$ matrices. We can identify both points in G and vectors in T_eG with matrices: for example if $a(t)$ is a curve of matrices in G with $\det a(t)\neq 0$ then its derivative is $\dot a(t)\in T_{a(t)}G$, which is just another matrix.

(a) Show that the left translation push-forward is $(L_a)_*(x) = ax$ for $a \in G$ and $x \in T_eG$.

The left translation is $L_a(g) = ag$ for $a, g \in G$. Now if $\gamma\colon (-\varepsilon, \varepsilon) \to G$ is any curve with $\gamma(0) = e$ and $\gamma'(0) = x$, then we have

$$(L_a)_*x = \frac{d}{dt}\Big|_{t=0} L_a(\gamma(t)) = \frac{d}{dt}\Big|_{t=0} a\gamma(t) = a\gamma'(0) = ax.$$

(b) Show that the flow of the left-invariant vector field X on G generated by $x \in T_eG$ is given by $\Phi(t, a) = ae^{tx}$.

Solution: First we figure out what X actually is. By definition of a left-invariant field, we have $X_g = (L_g)_*x = gx$ for every $g \in G$. Integral curves then satisfy the formula $\gamma'(t) = X_{\gamma(t)}$, or more explicitly $\gamma'(t) = \gamma(t)x$ where x is some matrix. The solution of this differential equation is $\gamma(t) = \gamma(0)e^{tx}$, so the flow is $\gamma(t) = \Phi(t, a) = ae^{tx}$.

(c) Use Problem 2 to compute the bracket $[X, Y]$ for left-invariant fields on G generated by matrices $x, y \in T_eG$. Show that it is the left-invariant vector field generated by $xy - yx \in T_eG$. (Hint: the Lie bracket of left-invariant fields is left-invariant by Proposition 14.2.9.)

Solution: If X is the left-invariant field generated by x, then the flow of X is $\Phi_t(a) = ae^{tx}$. Similarly the flow of Y is $\Psi_s(a) = ae^{sy}$. Thus by Problem 1 we have

$$[X, Y]_e = -\frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial}{\partial s}\Big|_{s=0} \Phi_t \circ \Psi_s \circ \Phi_{-t} \circ \Psi_{-s}(e).$$

Now we compute the term inside to get

$$\begin{aligned}\Phi_t \circ \Psi_s \circ \Phi_{-t} \circ \Psi_{-s}(e) = \Phi_t \circ \Psi_s \circ \Phi_{-t}(e^{-sy}) &= \Phi_t \circ \Psi_s(e^{-sy}e^{-tx}) \\ &= \Phi_t(e^{-sy}e^{-tx}e^{sy}) = e^{-sy}e^{-tx}e^{sy}e^{tx}.\end{aligned}$$

Taking the derivatives and using the formulas $\frac{d}{dt}e^{tx} = xe^{tx} = e^{tx}x$, we get

$$\begin{aligned}[X, Y]_e &= -\frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial}{\partial s}\Big|_{s=0} \Phi_t \circ \Psi_s \circ \Phi_{-t} \circ \Psi_{-s}(e) \\ &= -\frac{\partial}{\partial t}\Big|_{t=0} \Big(- e^{-sy}ye^{-tx}e^{sy}e^{tx} + e^{-sy}e^{-tx}ye^{sy}e^{tx}\Big)\Big|_{s=0} \\ &= -\frac{\partial}{\partial t}\Big|_{t=0} \big(- ye^{-tx}e^{tx} + e^{-tx}ye^{tx}\big) \\ &= -\frac{\partial}{\partial t}\Big|_{t=0} \big(- y + e^{-tx}ye^{tx}\big) \\ &= -\big(0 - e^{-tx}xye^{tx} + e^{-tx}yxe^{tx}\big)\Big|_{t=0} \\ &= -(-xy + yx) \\ &= xy - yx.\end{aligned}$$

4. Find a basis $\{e_1, e_2, e_3\}$ of traceless matrices for the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, and use the previous problem to compute all the nonzero Lie brackets $[e_i, e_j]$.

Solution: A simple basis is

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is enough to compute $[e_1, e_2]$, $[e_2, e_3]$, and $[e_3, e_1]$ by antisymmetry.
We get

$$[e_1, e_2] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = 2e_3,$$

and similarly

$$[e_2, e_3] = -2e_1, \qquad [e_3, e_1] = -2e_2.$$