Math 70900 Homework #7
due Friday, October 20

Read "Introduction to Differential Geometry" through Chapter 13.

1. On $M = \mathbb{R}^2$, let $X = y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y}$ and $Y = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}$.
   (a) Show that $[X, Y] = 0$.
   (b) Find the flows Φ_t of X and Ψ_t of Y.
   (c) Construct an explicit coordinate chart near $(1, 0)$ such that $X = \frac{\partial}{\partial u}$ and $Y = \frac{\partial}{\partial v}$.
2. Suppose X and Y are vector fields on M, with flows Φ_t and Ψ_t respectively. Prove that
$$[X, Y]_p = -\frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial}{\partial s}\Big|_{s=0} \Phi_t \circ \Psi_s \circ \Phi_{-t} \circ \Psi_{-s}(p).$$
3. Let $G = GL_n(\mathbb{R})$ be the Lie group of all real invertible $n \times n$ matrices. We can identify both points in G and vectors in T_eG with matrices: for example if $a(t)$ is a curve of matrices in G with $\det a(t) \neq 0$ then its derivative is $\dot{a}(t) \in T_{a(t)}G$, which is just another matrix.
   (a) Show that the left translation push-forward is $(L_a)_*(x) = ax$ for $a \in G$ and $x \in T_eG$.
   (b) Show that the flow of the left-invariant vector field X on G generated by $x \in T_eG$ is given by $\Phi(t, a) = ae^{tx}$.
   (c) Use Problem 1 to compute the bracket $[X, Y]$ for left-invariant fields on G generated by matrices $x, y \in T_eG$. Show that it is the left-invariant vector field generated by $xy - yx \in T_eG$. (Hint: the Lie bracket of left-invariant fields is left-invariant by Proposition 14.2.9.)
4. Find a basis $\{e_1, e_2, e_3\}$ of traceless matrices for the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, and use the previous problem to compute all the Lie brackets $[e_i, e_j]$.