Math 70900 Homework #7 Solutions

1. Prove that if M is a smooth manifold with a trivial tangent bundle, then M must be orientable.

   Solution: Let $\Upsilon\colon TM \to M \times \mathbb{R}^n$ be a trivialization. We want to prove that M is orientable, i.e., that there is a family of coordinate charts (ϕ_i, U_i) covering M such that $\det D(\phi_i \circ \phi_j^{-1}) > 0$ always.

   Now use Υ to define n linearly-independent vector fields $V_1, \ldots, V_n$ on M. For any given coordinate chart (ϕ, U) on M, write these vector fields as

$$V_j = \sum_{k=1}^n c_j^k \frac{\partial}{\partial x^k}.$$

   Then c_j^k is a nonsingular matrix since both $\{\frac{\partial}{\partial x^k}\}$ and $\{X_j\}$ are linearly independent. Hence the determinant $\det C$ of this matrix is either positive or negative, and by continuity of the vector fields this determinant must also be continuous. Hence it's *always* positive or *always* negative.

   If the determinant $\det C$ is negative, then we can change the coordinate chart to a new chart with the same domain by $y^1 = x^1$, $\ldots$, $y^{n-1} = x^{n-1}$, $y^n = -x^n$, and in this chart we will have $\frac{\partial}{\partial x^k} = \frac{\partial}{\partial y^k}$ when $k \neq n$ and $\frac{\partial}{\partial x^n} = -\frac{\partial}{\partial y^n}$. Hence the corresponding matrix $\tilde{C}$ in the y-basis will be the same as the matrix C in the x-basis except that the last row or column (depending how you write it) will flip sign. Thus the determinant will also flip sign.

   By doing the above we can take any given atlas and flip the sign of the last coordinate so that the determinant of the matrix C that expresses the fields $\{V_j\}$ in the coordinate basis is always positive. So suppose we have done this.

   Now let $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ be two coordinate charts, and suppose

$$V_j = \sum_{k=1}^n c_j^k \frac{\partial}{\partial x^k} \qquad \text{and} \qquad V_j = \sum_{\ell=1}^n b_j^\ell \frac{\partial}{\partial y^\ell}.$$

   Then using the change-of-basis formula for coordinate vectors, we have

$$\sum_{k=1}^n c_j^k \sum_{\ell=1}^n \frac{\partial y^\ell}{\partial x^k} \frac{\partial}{\partial y^\ell} = \sum_{\ell=1}^n b_\ell^j \frac{\partial}{\partial y^\ell},$$

   which obviously implies that

$$b_\ell^j = \sum_{k=1}^n c_j^k \frac{\partial y^\ell}{\partial x^k}$$

   for every j and ℓ. This is just a product of matrices: we have $\det B = \det C \det \frac{\partial \mathbf{y}}{\partial \mathbf{x}}$. Since $\det B > 0$ and $\det C > 0$ by construction, we conclude that $\det \frac{\partial \mathbf{y}}{\partial \mathbf{x}} > 0$ for all of the charts in our atlas. This is exactly the definition of orientability.

2. Suppose M is a smooth submanifold of a smooth manifold N in the sense of Definition 9.1.9, and let $\iota\colon M \to N$ be the inclusion. Let Y be a smooth vector field on N such that whenever $p \in M$ we have $Y(\iota(p)) \in \iota_*[T_pM]$. Prove that there is a unique smooth vector field X on M such that $\iota_* \circ X = Y \circ \iota$. (Hint: for each $p \in M$ the vector $X(p)$ is completely determined; what does it look like in a coordinate chart?)

   Solution: Since M is a smooth submanifold, around every point $p \in M$ there is a coordinate chart (ψ, V) on N with $\iota(p) \in V$ and $\psi[M \cap V] = \{(x^1, \ldots, x^m, 0, \ldots, 0) \in \mathbb{R}^n\}$. Let $U = M \cap V$, an open set in M, and let $\phi\colon U \to \mathbb{R}^m$ be the map $\phi(p) = \pi_{\mathbb{R}^m}(\psi(\iota(p)))$ which projects an n-tuple onto its first m components. Then ϕ is a coordinate chart on M, and in the charts (ϕ, U) and (ψ, V), the smooth map ι takes the form

$$(x^1, \ldots, x^n) = \psi \circ \iota \circ \phi^{-1}(u^1, \ldots, u^m) = (u^1, \ldots, u^m, 0, \ldots, 0).$$

   As a result the formula for ι_* is

$$\iota_*\left(\frac{\partial}{\partial u^k}\bigg|_p\right) = \frac{\partial}{\partial x^k}\bigg|_{\iota(p)} \qquad \text{for } 1 \le k \le m.$$

   Now let Y be a smooth vector field on N. By definition Y is smooth if in every coordinate chart it is smooth. Let (ψ, V) be the chart above and let (Ψ, TV) be the corresponding chart on TN. Then

$$Y(q) = \sum_{k=1}^{n} y^k(q)\, \frac{\partial}{\partial x^k}\bigg|_q$$

   for some functions $y^k(q)$, and in coordinates we have

$$\Psi \circ Y \circ \psi^{-1}(x^1, \ldots, x^n) = \big(x^1, \ldots, x^n, b^1(x^1, \ldots, x^n), \ldots, b^n(x^1, \ldots, x^n)\big),$$

   where $b^k = y^k \circ \psi^{-1}$. So Y is smooth if and only if the functions b^k are smooth on $\mathbb{R}^n$. Now since

$$\Psi \circ \iota_* \circ \Phi^{-1}(u^1, \ldots, u^m, a^1, \ldots, a^m) = (u^1, \ldots, u^m, 0, \ldots, 0, a^1, \ldots, a^m, 0, \ldots, 0)$$

   and

$$\Psi \circ Y \circ \iota \circ \phi^{-1}(u^1, \ldots, u^m) = \big(u^1, \ldots, u^m, 0, \ldots, 0, b^1(u^1, \ldots, u^m, 0, \ldots, 0), \ldots, b^n(u^1, \ldots, u^m, 0, \ldots, 0)\big),$$

   we see that $Y(\iota(p)) \in \iota_*[T_pM]$ if and only if in coordinates we have

$$b^k(u^1, \ldots, u^m, 0, \ldots, 0) = 0 \qquad \text{for } m+1 \le k \le n$$

   for all $(u^1, \ldots, u^m) \in \mathbb{R}^m$.

   If $Y(\iota(p)) \in \iota_*[T_pM]$ for a $p \in M$, then $Y(\iota(p)) = \iota_*(X(p))$ for some $X(p) \in T_pM$, and $X(p)$ is unique since ι_* is one-to-one. This works for every $p \in M$ and thus defines a map from M to TM with $X(p) \in T_pM$ for all p. We just have to check that it's smooth. In coordinates (ϕ, U) and (Φ, TU) on M and TM as above, we must have

$$\Phi \circ X \circ \phi^{-1}(u^1, \ldots, u^m) = \big(u^1, \ldots, u^m, b^1(u^1, \ldots, u^m, 0, \ldots, 0), \ldots, b^m(u^1, \ldots, u^m, 0, \ldots, 0)\big),$$

and since each b^k is smooth on all of $\mathbb{R}^n$, it is in particular smooth on $\mathbb{R}^m \times \{\mathbf{0}\} \subset \mathbb{R}^n$. Thus since X is smooth in coordinates, it is a smooth vector field.

3. Consider $\mathbb{R}^4$ as the quaternions, where each element is written as $(w, x, y, z) = q = w + xi + yj + zk$, with the multiplication defined to be a bilinear operation satisfying

$$i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ij = k, jk = i, ki = j, \quad \text{and} \quad ji = -k, kj = -i, ik = -j.$$

(You can take for granted that this multiplication is well-defined and associative.) Show that the unit quaternions (those satisfying $w^2 + x^2 + y^2 + z^2 = 1$) are a group, which is identified with S^3. Compute the left-invariant vector fields on S^3. (Hint: first compute the left-invariant fields on $\mathbb{R}^4$, then use the previous problem.)

Solution: First we check the group property, which is the statement that if $P = w + xi + yj + zk$ and $Q = a + bi + cj + dk$ both have unit length, then so does $QP = L_Q(P)$. Let $Q = (a, b, c, d) = a + bi + cj + dk$ be a fixed quaternion; then we compute for a quaternion $P = w + xi + yj + zk$ that

$$\begin{aligned} L_Q(P) &= (a + bi + cj + dk)(w + xi + yj + zk) \\ &= (aw - bx - cy - dz) + (bw + ax + cz - dy)i \\ &\qquad + (ay + cw + dx - bz)j + (az + dw + by - cx)k. \end{aligned}$$

Or in coordinates we have

$$\begin{aligned} (q, r, s, t) &= L_{(a,b,c,d)}(w, x, y, z) \\ &= (aw - bx - cy - dz, bw + ax + cz - dy, ay + cw + dx - bz, az + dw + by - cx). \end{aligned} \tag{1}$$

By expanding the squared length of this, we get

$$\begin{aligned} &(aw-bx-cy-dz)^2+(bw+ax+cz-dy)^2+(ay+cw+dx-bz)^2+(az+dw+by-cx)^2 = \\ &(a^2+b^2+c^2+d^2)w^2 + (b^2+a^2+d^2+c^2)x^2 + (c^2+d^2+a^2+b^2)y^2 + (d^2+c^2+b^2+a^2)z^2 \\ &\quad + 2wx(-ab+ab+cd-cd) + 2wy(-ac-bd+ac+bd) + 2wz(-ad+bc-bcad) \\ &\quad + 2xy(bc-ad+ad-bc) + 2xz(bd+ac-bd-ac) + 2yz(cd-cd-ab+ab) \\ &\qquad = 1 \cdot (w^2+x^2+y^2+z^2) = 1. \end{aligned}$$

So indeed, the product of unit quaternions is still unit, and the unit quaternions form a group.

Using formula (1) for the left-translation, the derivative map is computed by

$$\begin{aligned}
(L_Q)_*\left(\tfrac{\partial}{\partial w}\Big|_P\right) &= \tfrac{\partial q}{\partial w}\tfrac{\partial}{\partial q}\Big|_{QP} + \tfrac{\partial r}{\partial w}\tfrac{\partial}{\partial r}\Big|_{QP} + \tfrac{\partial s}{\partial w}\tfrac{\partial}{\partial s}\Big|_{QP} + \tfrac{\partial t}{\partial w}\tfrac{\partial}{\partial t}\Big|_{QP} \\
&= a\,\tfrac{\partial}{\partial q}\Big|_{QP} + b\,\tfrac{\partial}{\partial r}\Big|_{QP} + c\,\tfrac{\partial}{\partial s}\Big|_{QP} + d\,\tfrac{\partial}{\partial t}\Big|_{QP} \\
(L_Q)_*\left(\tfrac{\partial}{\partial x}\Big|_P\right) &= -b\,\tfrac{\partial}{\partial q}\Big|_{QP} + a\,\tfrac{\partial}{\partial r}\Big|_{QP} + d\,\tfrac{\partial}{\partial s}\Big|_{QP} - c\,\tfrac{\partial}{\partial t}\Big|_{QP} \\
(L_Q)_*\left(\tfrac{\partial}{\partial y}\Big|_P\right) &= -c\,\tfrac{\partial}{\partial q}\Big|_{QP} - d\,\tfrac{\partial}{\partial r}\Big|_{QP} + a\,\tfrac{\partial}{\partial s}\Big|_{QP} + b\,\tfrac{\partial}{\partial t}\Big|_{QP} \\
(L_Q)_*\left(\tfrac{\partial}{\partial z}\Big|_P\right) &= -d\,\tfrac{\partial}{\partial q}\Big|_{QP} + c\,\tfrac{\partial}{\partial r}\Big|_{QP} - b\,\tfrac{\partial}{\partial s}\Big|_{QP} + a\,\tfrac{\partial}{\partial t}\Big|_{QP}.
\end{aligned}$$

Now set $P = e = (1,0,0,0)$, the identity of S^3 (or of $\mathbb{R}^4$). The basis vectors here are spanned by $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$, and the three left-invariant fields are given at each $Q = a + bi + cj + dk$ by

$$\begin{aligned}
E_1\big|_Q &= (L_Q)_*\left(\tfrac{\partial}{\partial x}\big|_e\right) = -b\,\tfrac{\partial}{\partial q}\Big|_Q + a\,\tfrac{\partial}{\partial r}\Big|_Q + d\,\tfrac{\partial}{\partial s}\Big|_Q - c\,\tfrac{\partial}{\partial t}\Big|_Q \\
E_2\big|_Q &= (L_Q)_*\left(\tfrac{\partial}{\partial y}\big|_e\right) = -c\,\tfrac{\partial}{\partial q}\Big|_Q - d\,\tfrac{\partial}{\partial r}\Big|_Q + a\,\tfrac{\partial}{\partial s}\Big|_Q + b\,\tfrac{\partial}{\partial t}\Big|_Q \\
E_3\big|_Q &= (L_Q)_*\left(\tfrac{\partial}{\partial z}\big|_e\right) = -d\,\tfrac{\partial}{\partial q}\Big|_Q + c\,\tfrac{\partial}{\partial r}\Big|_Q - b\,\tfrac{\partial}{\partial s}\Big|_Q + a\,\tfrac{\partial}{\partial t}\Big|_Q.
\end{aligned}$$

Now renaming both (q,r,s,t) and (a,b,c,d) to standard coordinates (w,x,y,z), we get the form

$$\begin{aligned}
E_1 &= -x\,\tfrac{\partial}{\partial w} + w\,\tfrac{\partial}{\partial x} + z\,\tfrac{\partial}{\partial y} - y\,\tfrac{\partial}{\partial z} \\
E_2 &= -y\,\tfrac{\partial}{\partial w} - z\,\tfrac{\partial}{\partial x} + w\,\tfrac{\partial}{\partial y} + x\,\tfrac{\partial}{\partial z} \\
E_3 &= -z\,\tfrac{\partial}{\partial w} + y\,\tfrac{\partial}{\partial x} - x\,\tfrac{\partial}{\partial y} + w\,\tfrac{\partial}{\partial z}.
\end{aligned}$$

All of these are vector fields in $\mathbb{R}^4$ that are orthogonal to each other and to the unit normal vector field

$$E_0 = w\,\tfrac{\partial}{\partial w} + x\,\tfrac{\partial}{\partial x} + y\,\tfrac{\partial}{\partial y} + z\,\tfrac{\partial}{\partial z},$$

so they are all tangent to S^3 and induce vector fields on it as in problem 2.

4. Suppose M is a smooth manifold with a discrete group G which gives a free and proper action on M by smooth maps $\phi_g\colon M \to M$ for each $g \in G$, so that $K = M/G$ is a smooth manifold by Theorem 9.1.8. Let $\tau\colon M \to K$ be the quotient map.

   (a) If X is a vector field on M, and if $(\phi_g)_*X(p) = X(\phi_g(p))$ for every $g \in G$, prove that there is a vector field Y on K such that $\tau_* \circ X(p) = Y(\tau(p))$ for all $p \in X$. We say "X descends to K."

   Solution: Given $q \in K$, pick a $p \in \tau^{-1}\{q\}$ and push $X(p)$ forward to $q = \tau(p)$ to get $Y(q) = \tau_*(X(p))$. We want to show that this does not depend on the choice of $p \in \tau^{-1}\{q\}$. If p' is also in $\tau^{-1}\{q\}$, then $p' = \phi_g(p)$ for some $g \in G$. Now

$X(p') = X(\phi_g(p)) = (\phi_g)_* X(p)$ by assumption. Furthermore since $\tau \circ \phi_g = \tau$ by definition of the quotient map, we have $\tau_* \circ (\phi_g)_* = \tau_*$ by the Chain Rule, and thus $\tau_*(X(p')) = \tau_*(\phi_g)_*(X(p)) = \tau_*(X(p))$. So $\tau_*(X(p))$ is uniquely determined by $\tau(p)$.

We have now defined Y on K and we need to prove it is smooth. This is easy to do in the smooth structure we defined for K in Theorem 9.1.8. Around any point $p \in M$ we take a coordinate chart (ϕ, U) where U is small enough that $\phi_g[U] \cap U = \emptyset$ for any $g \neq e$. Then $\tau|_U$ is a diffeomorphism from U to $\tau[U]$, and coordinates on $\pi[U]$ are given by $\phi \circ \tau|_U^{-1}$. In such coordinates, τ is basically the identity: $\tau(x^1, \ldots, x^n) = (y^1, \ldots, y^n)$. Thus if $X(p) = \sum_i a^i(p) \frac{\partial}{\partial x^i}\big|_p$, then we have $Y(q) = \sum_i a^i(\tau(p)) \frac{\partial}{\partial y^i}\big|_q$. So Y is just as smooth as X.

(b) In the special case where $M = \mathbb{R}^2$ and the group G is the group of isometries generated by $g_1(x,y) = (x+1, y)$ and $g_2(x,y) = (-x, y+1)$, show that $Y(p) = \frac{\partial}{\partial y}\big|_p$ descends to K but that $X(p) = \frac{\partial}{\partial x}\big|_p$ does not descend to K.

Solution: We just need to check that $(g_1)_* Y = Y \circ g_1$ and that $(g_2)_* Y = Y \circ g_2$. Notice that

$$(g_1)_* \left(\frac{\partial}{\partial x}\bigg|_{(x,y)} \right) = \frac{\partial}{\partial x}\bigg|_{(x+1,y)} \quad \text{and} \quad (g_1)_* \left(\frac{\partial}{\partial y}\bigg|_{(x,y)} \right) = \frac{\partial}{\partial y}\bigg|_{(x+1,y)}$$
$$(g_2)_* \left(\frac{\partial}{\partial x}\bigg|_{(x,y)} \right) = -\frac{\partial}{\partial x}\bigg|_{(-x,y+1)} \quad \text{and} \quad (g_2)_* \left(\frac{\partial}{\partial y}\bigg|_{(x,y)} \right) = \frac{\partial}{\partial y}\bigg|_{(-x,y+1)}.$$

Hence $(g_1)_*(Y(x,y)) = Y(g_1(x,y))$ and $(g_2)_*(Y(x,y)) = Y(g_2(x,y))$, so that Y descends. But $(g_1)_*(X(x,y)) = X(g_1(x,y))$ while $(g_2)_*(X(x,y)) = -X(g_2(x,y))$, so X does not descend.

(c) Show directly that there cannot be any other vector field $Z = h(x,y) \frac{\partial}{\partial x}\big|_{(x,y)} + j(x,y) \frac{\partial}{\partial y}\big|_{(x,y)}$ that is linearly independent of Y everywhere and descends to K.

Solution: Write $Z = f(x,y) \frac{\partial}{\partial x}\big|_{(x,y)} + g(x,y) \frac{\partial}{\partial y}\big|_{(x,y)}$ for some smooth functions f and g. Clearly Z is linearly independent of Y everywhere if and only if $f(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}^2$. Since $\mathbb{R}^2$ is connected, the only way this is possible is if f is either always positive, or always negative (for if it were both, then by the Intermediate Value Theorem along a path, f would have to be zero somewhere in between).

In order to have $(g_1)_* Z = Z \circ g_1$ we must have $f(x,y) = f(x+1,y)$ and $g(x,y) = g(x+1,y)$ for all $(x,y) \in \mathbb{R}^2$. Furthermore to have $(g_2)_* Z = Z \circ g_2$ we must have $-f(x,y) = f(x,y+1)$ and $g(x,y) = g(x,y+1)$ for all $(x,y) \in \mathbb{R}^2$. Now $f(x,y+1) = -f(x,y)$ contradicts the fact that f is always of the same sign and never zero. So there is no such Z.

(d) Prove that the Klein bottle does not have trivial tangent bundle.

Solution: If the tangent bundle were trivial, then we could have two vector fields on K that are everywhere linearly independent. But the argument above shows that we only have one vector field that is nowhere zero, and no other field can be chosen to give a linearly independent one.

5. The following steps are used to construct a partition of unity on a noncompact manifold.

(a) Given a noncompact manifold M (which is Hausdorff and second-countable), show that there is a countable collection of open subsets V_i such that

- the closure $\overline{V_i}$ of V_i is a compact subset of M
- $\overline{V_i} \subset V_{i+1}$ for each i
- $\bigcup_{i=1}^{\infty} V_i = M$

Solution: Cover M with coordinate charts. Since M is second-countable, it is Lindelöf, and thus we only need countably many charts to cover. Call them (ϕ_i, U_i) for $i \in \mathbb{N}$.

The idea is to set up V_i as the union of finitely many coordinate balls, where the number of balls and their radii increases simultaneously. Precisely, we let

$$V_i = \bigcup_{j=1}^{i} \phi_j^{-1}[B_i(\mathbf{0})].$$

The closure of a finite union is the finite union of the closure, and since the ϕ_j are homeomorphisms, we have

$$\overline{V_i} = \bigcup_{j=1}^{i} \phi_j^{-1}[\overline{B_i(\mathbf{0})}].$$

The ball of radius i is compact, and thus so is $\overline{V_i}$. Secondly we have

$$\overline{V_i} = \bigcup_{j=1}^{i} \phi_j^{-1}[\overline{B_i(\mathbf{0})}] \subset \bigcup_{j=1}^{i} \phi_j^{-1}[B_{i+1}(\mathbf{0})] \subset \bigcup_{j=1}^{i+1} \phi_j^{-1}[B_{i+1}(\mathbf{0})] = V_{i+1}.$$

Finally, for any point $p \in M$, we know that $p \in U_k$ for some $k \in \mathbb{N}$. Furthermore the Euclidean distance from $\phi_k(p)$ to $\mathbf{0}$ is less than some positive integer ℓ. Hence if $m = \max\{k, \ell\}$, then

$$p \in \phi_k^{-1}[B_\ell(\mathbf{0})] \subset \phi_k^{-1}[B_m(\mathbf{0})] \subset V_m.$$

Thus the family $\{V_m : m \in \mathbb{N}\}$ covers M.

(b) If $\{V_i\}$ is a collection as above and $W_i = V_i \backslash \overline{V_{i-2}}$, show that each W_i intersects only W_{i-1} and W_{i+1}.

Solution: The only thing that matters here is that $V_i \subset \overline{V_i} \subset V_{i+1}$. Consider $W_i \cap W_j$ with $j > i$. Then

$$W_i \cap W_j = (V_i \backslash \overline{V_{i-2}}) \cap (V_j \backslash \overline{V_{j-2}}) = V_i \cap \overline{V_{i-2}}^c \cap V_j \cap \overline{V_{j-2}}^c.$$

Now $V_i \subset V_j$ and $\overline{V_{i-2}} \subset \overline{V_{j-2}}$, so that $\overline{V_{j-2}}^c \subset \overline{V_{i-2}}^c$. Thus

$$W_i \cap W_j \subset V_i \cap \overline{V_{j-2}}^c.$$

If $j - 2 \geq i$ then $\overline{V_{j-2}}^c \subset \overline{V}_i^c \subset V_i^c$, which means $W_i \cap W_j \subset V_i \cap V_i^c = \emptyset$. Hence the only way $W_i \cap W_j \neq \emptyset$ with $j > i$ is if $j = i + 1$.