Math 70900 Homework #7
due Friday, October 20

Read "Introduction to Differential Geometry" through Chapter 13.

1. Prove that if M is a smooth manifold with a trivial tangent bundle, then M must be orientable.

2. Suppose M is a smooth submanifold of a smooth manifold N in the sense of Definition 9.1.9, and let $\iota\colon M \to N$ be the inclusion. Let Y be a smooth vector field on N such that whenever $p \in M$ we have $Y(\iota(p)) \in \iota_*[T_pM]$. Prove that there is a unique smooth vector field X on M such that $\iota_* \circ X = Y \circ \iota$. (Hint: for each $p \in M$ the vector $X(p)$ is completely determined; what does it look like in a coordinate chart?)

3. Consider $\mathbb{R}^4$ as the quaternions, where each element is written as $(w, x, y, z) = q = w + xi + yj + zk$, with the multiplication defined to be a bilinear operation satisfying
$$i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ij = k, jk = i, ki = j, \quad \text{and} \quad ji = -k, kj = -i, ik = -j.$$
(You can take for granted that this multiplication is well-defined and associative.) Show that the unit quaternions (those satisfying $w^2 + x^2 + y^2 + z^2 = 1$) are a group, which is identified with S^3. Compute the left-invariant vector fields on S^3. (Hint: first compute the left-invariant fields on $\mathbb{R}^4$, then use the previous problem.)

4. Suppose M is a smooth manifold with a discrete group G which gives a free and proper action on M by smooth maps $\phi_g\colon M \to M$ for each $g \in G$, so that $K = M/G$ is a smooth manifold by Theorem 9.1.8. Let $\tau\colon M \to K$ be the quotient map.
   (a) If X is a vector field on M, and if $(\phi_g)_* X(p) = X(\phi_g(p))$ for every $g \in G$, prove that there is a vector field Y on K such that $\tau_* \circ X(p) = Y(\tau(p))$ for all $p \in X$. We say "X descends to K."
   (b) In the special case where $M = \mathbb{R}^2$ and the group G is the group of isometries generated by $g_1(x, y) = (x+1, y)$ and $g_2(x, y) = (-x, y+1)$, show that $Y(p) = \frac{\partial}{\partial y}\big|_p$ descends to K but that $X(p) = \frac{\partial}{\partial x}\big|_p$ does not descend to K.
   (c) Show directly that there cannot be any other vector field $Z = h(x, y)\,\frac{\partial}{\partial x}\big|_{(x,y)} + j(x, y)\,\frac{\partial}{\partial y}\big|_{(x,y)}$ that is linearly independent of Y everywhere and descends to K.
   (d) Prove that the Klein bottle does not have trivial tangent bundle.

5. The following steps are used to construct a partition of unity on a noncompact manifold.
   (a) Given a noncompact manifold M (which is Hausdorff and second-countable), show that there is a countable collection of open subsets V_i such that
      - the closure $\overline{V_i}$ of V_i is a compact subset of M
      - $\overline{V_i} \subset V_{i+1}$ for each i
      - $\bigcup_{i=1}^{\infty} V_i = M$
   (b) If $\{V_i\}$ is a collection as above and $W_i = V_i \backslash \overline{V_{i-2}}$, show that each W_i intersects only W_{i-1} and W_{i+1}.