Math 70900 Homework #6 Solutions

1. (Alternate approach to vectors.) For $p \in M$, let $\mathcal{G}_p$ denote the set of all germs of C^∞ functions at p (that is, smooth real-valued functions defined on some open neighborhood of p under the equivalence relation that functions are equal if they coincide on some neighborhood of p which is contained in both their domains; see Remark 10.3.3). Define a *schmector* to be a linear operator D from the algebra $\mathcal{G}_p$ to $\mathbb{R}$ which satisfies the Leibniz rule $D(f \cdot g) = g(p)D(f) + f(p)D(g)$.

   (a) Prove that addition and multiplication are well-defined operations on germs (so that it is in fact an algebra), and that for any $v \in T_pM$ the operator $D_v \colon f \mapsto v(f)$ is a schmector.

   Solution: If $F = [(f, U)]$ and $G = [(g, V)]$, then to prove that $F + G$ is well-defined, we want to check that if $(f, U) \equiv (h, A)$ and $(g, V) \equiv (j, B)$ then $(f + g, U \cap V) \equiv (h + j, A \cap B)$. Now $f = h$ on some open $C \subset A \cap U$ and $g = j$ on some open $D \subset B \cap V$, which means that $f + g = h + j$ on the set $C \cap D \subset (A \cap U) \cap (B \cap V) = (U \cap V) \cap (A \cap B)$. Hence $(f + g, U \cap V) \equiv (h + j, A \cap B)$. Similarly multiplication is well-defined.

   Now we show that D_v is a schmector. We already know that D_v is a linear operator, and we just need to show the Leibniz rule. Let γ be any representative of v. Then

$$\begin{aligned} v(f \cdot g) = \frac{d}{dt}\Big(f(\gamma(t))g(\gamma(t))\Big)\Big|_{t=0} &= f(\gamma(0))\,\frac{d}{dt}g(\gamma(t))\Big|_{t=0} + g(\gamma(0))\,\frac{d}{dt}f(\gamma(t))\Big|_{t=0} \\ &= f(p)v(g) + g(p)v(f) \end{aligned}$$

   by the usual product rule for functions.

   (b) Prove using the Leibniz rule that if $f \colon U \to \mathbb{R}$ is a constant function for some open set $U \ni p$, then $D(f) = 0$ for any schmector D.

   Solution: Suppose f is a constant function, $f(p) = c$ for all $p \in M$. Let h be the constant function $h(p) = 1$ for all $p \in M$. Then $f = ch$ and $D_v(f) = cD_v(h)$. On the other hand, $h^2 = h$, which means that

$$D_v(h) = h(p)D_v(h) + h(p)D_v(h) = 2D_v(h).$$

   Hence we must have $D_v(h) = 0$ and thus also $D_v(f) = 0$.

   (c) Prove the following Lemma: for any C^∞ function $g \colon \mathbb{R}^n \to \mathbb{R}$, we can write

$$g(\mathbf{x}) = a + \sum_{k=1}^n b_k x^k + \sum_{i,j=1}^n c_{ij}(\mathbf{x}) x^i x^j,$$

   where $a = g(\mathbf{0})$, $b_k = \dfrac{\partial g}{\partial x^k}(\mathbf{0})$, and $c_{ij}(\mathbf{x}) = \displaystyle\int_0^1 (1-t)\frac{\partial^2 g}{\partial x^i \partial x^j}(t\mathbf{x})\,dt$. (Hint: Suppose for a fixed $\mathbf{x}$ we denote $h(t) = g(tx^1, \dots, tx^n)$; show that $h''(t) = \displaystyle\sum_{i,j=1}^n x^i x^j \frac{\partial^2 g}{\partial x^i \partial x^j}(t\mathbf{x})$. Then use integration by parts.)

Solution: Let's check the formula in the hint: we have

$$h'(t) = \sum_{j=1}^{n} \frac{\partial g}{\partial x^j}(tx^1, \ldots, tx^n)\, \frac{d}{dt}(tx^j) = \sum_{j=1}^{n} x^j\, \frac{\partial g}{\partial x^j}(tx^1, \ldots, tx^n)$$

$$h''(t) = \sum_{i,j=1}^{n} x^j\, \frac{\partial^2 g}{\partial x^i \partial x^j}(tx^1, \ldots, tx^n)\, \frac{d}{dt}(tx^i) = \sum_{i,j=1}^{n} x^i x^j\, \frac{\partial^2 g}{\partial x^i \partial x^j}(tx^1, \ldots, tx^n).$$

We therefore have

$$\begin{aligned}\sum_{i,j=1}^{n} c_{ij}(\mathbf{x}) x^i x^j &= \sum_{i,j=1}^{n} x^i x^j \int_0^1 (1-t) \frac{\partial^2 g}{\partial x^i \partial x^j}(t\mathbf{x})\, dt = \int_0^1 (1-t) h''(t)\, dt \\ &= (1-t)h'(t)\Big|_{t=0}^{t=1} + \int_0^1 h'(t)\, dt = -h'(0) + h(1) - h(0).\end{aligned}$$

We thus have

$$h(1) = h(0) + h'(0) + \sum_{i,j=1}^{n} c_{ij}(\mathbf{x}) x^i x^j, \text{ which translates into}$$

$$g(\mathbf{x}) = g(\mathbf{0}) + \sum_{j=1}^{n} x^j \frac{\partial g}{\partial x^j}(\mathbf{0}) + \sum_{i,j=1}^{n} c_{ij}(\mathbf{x}) x^i x^j.$$

(d) Use the previous result and the Leibniz rule to prove that the space of all schmectors is an n-dimensional vector space, and that the identification $v \mapsto D_v$ is an isomorphism from tangent vectors to schmectors.

Solution: For any schmector D on $\mathcal{G}_p$ and any germ of a function f at p, we have

$$\begin{aligned} D(f) &= D\left(a + \sum_{k=1}^{n} b_k \phi^k(q) + \sum_{i,j=1}^{n} C_{ij}(q)\phi^i(q)\phi^j(q) \right) \\ &= aD(1) + \sum_{k=1}^{n} b_k D(\phi^k) + \sum_{i,j=1}^{n} D(C_{ij}\phi^i\phi^j) \\ &= a \cdot 0 + \sum_{k=1}^{n} b_k D(\phi^k) + \sum_{i,j=1}^{n} \big(\phi^i(p) D(C_{ij}\phi^j) + C_{ij}(p)\phi^j(p) D(\phi^i)\big) \\ &= \sum_{k=1}^{n} b_k D(\phi^k), \end{aligned}$$

since $\phi^i(p) = \phi^j(p) = 0$ for all i, j.

Now define a set of schmectors $\{D_1, \ldots, D_n\}$ by $D_k = \frac{\partial}{\partial x^k}\big|_p$. Since $b_k = \frac{\partial}{\partial x^k}(f \circ \mathbf{x}^{-1})\big|_{\mathbf{0}}$, we know that $D_k(f) = b_k$. Hence the general formula above says that

$$D(f) = \sum_{k=1}^{n} D(\phi^k) D_k(f)$$

for every function f, which means that $D = \sum_{k=1}^n D(\phi^k) D_k$. Hence the set $\{D_1, \dots, D_n\}$ spans the vector space of all schmectors. Furthermore if $D = \sum_{k=1}^n a^k D_k = 0$ for some numbers $\{a_1, \dots, a_n\}$ then $D(\phi^k) = a^k = 0$ for all k; thus the set $\{D_1, \dots, D_n\}$ is a basis for the space of schmectors, and so the schmectors are an n-dimensional vector space.

The fact that $v \mapsto D_v$ is an isomorphism comes from the fact that it is linear by problem #4d and that a basis $\{\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p\}$ for T_pM immediately gives a basis $\{D_1, \dots, D_n\}$ for the schmectors.

2. A *Lie group* G is a smooth manifold such that

- G is a group under some multiplication operation,
- the inversion $F\colon G \to G$ given by $F(g) = g^{-1}$ is a smooth map, and
- the multiplication $P\colon G \times G \to G$ given by $P(g, h) = g \cdot h$ is a smooth map.

(a) Show that for any fixed g the left-translation $L_g\colon G \to G$ given by $L_g(p) = g \cdot p$ is a diffeomorphism.

Solution: For a fixed g, let $\iota\colon G \to G \times G$ be the map $\iota(p) = (g, p)$. Then ι is smooth by definition of the product manifold structure, and we have $L_g = P \circ \iota$ which is the composition of smooth maps. Thus L_g is smooth. To prove L_g is a diffeomorphism, note that we have

$$L_{g^{-1}} \circ L_g(p) = g^{-1} \cdot (g \cdot p) = (g^{-1} \cdot g) \cdot p = e \cdot p = p$$

for every $p \in G$. Hence $L_{g^{-1}} = L_g^{-1}$. Since $L_{g^{-1}}$ is also smooth, we see that L_g is a diffeomorphism.

(b) If $e \in G$ is the identity and $v \in T_eG$ is any vector, show that $X_v(g) = (L_g)_*(v)$ defines a smooth vector field X_v on G. (Such a vector field is called "left-invariant.")

Solution: The trick here is that we are differentiating with respect to the p variable in $g \cdot p$ while holding g fixed, then fixing $p = e$, then considering the result as g varies. So we really need to use the smoothness of $P\colon G \times G \to G$, not just the smoothness of L_g.

Let $g_o \in G$ be a particular point; let (ψ, V) be a coordinate neighborhood of g_o and let (ϕ, U) be a coordinate neighborhood of e. (We can assume that $\phi(e) = \mathbf{0}$.) Then $(\psi \times \phi, V \times U)$ is a coordinate neighborhood of $G \times G$, and we can write

$$(z^1, \dots, z^n) = \psi \circ P \circ (\psi \times \phi)^{-1}(y^1, \dots, y^n, x^1, \dots, x^n) = \big(\rho^1(\mathbf{y}, \mathbf{x}), \dots, \rho^n(\mathbf{y}, \mathbf{x})\big),$$

where each ρ^k is smooth on $\mathbb{R}^n \times \mathbb{R}^n$. Then for each particular $g \in V$ with $\psi(g) = (y_o^1, \dots, y_o^n)$, we have

$$(z^1, \dots, z^n) = \psi \circ L_g \circ \phi^{-1}(x^1, \dots, x^n) = \big(\lambda^1(x^1, \dots, x^n), \dots, \lambda^n(x^1, \dots, x^n)\big),$$

where $\lambda^k(x^1, \dots, x^n) = \rho^k(y_o^1, \dots, y_o^n, x^1, \dots, x^n)$ for each k.

The push-forward is therefore given by

$$(L_g)_*\left(\sum_{i=1}^n a^i \frac{\partial}{\partial x^i}\Big|_e\right) = \sum_{i=1}^n\sum_{j=1}^n a^i \frac{\partial \lambda^j}{\partial x^i}\Big|_{\phi(e)} \frac{\partial}{\partial z^j}\Big|_g,$$

for any constants a^i, $1 \le i \le n$. For a fixed vector $\mathbf{a} = \sum_i a^i \frac{\partial}{\partial x^i}\big|_e \in T_eG$ we have, in coordinates (Ψ, TV), that $(L_g)_*(\mathbf{a})$ takes the form

$$\Psi\big((L_g)_*(\mathbf{a})\big) = \left(\psi(L_g(e)), \sum_i a^i \frac{\partial \lambda^1}{\partial x^i}\Big|_{\phi(e)}, \sum_i a^i \frac{\partial \lambda^n}{\partial x^i}\Big|_{\phi(e)}\right).$$

But recall that $L_g(e) = g$, that $\psi(g) = (y_o^1, \dots, y_o^n)$, and that $\lambda^j(x^1, \dots, x^n) = \rho^j(y_o^1, \dots, y_o^n, x^1, \dots, x^n)$. Since in addition we have $\phi(e) = \mathbf{0}$, we then obtain

$$\Psi\big((L_g)_*(\mathbf{a})\big) = \left(y_o^1, \dots, y_o^n, \sum_i a^i \frac{\partial \rho^1}{\partial x^i}(y_o^1, \dots, y_o^n, 0, \dots, 0), \dots, \sum_i a^i \frac{\partial \rho^n}{\partial x^i}(y_o^1, \dots, y_o^n, 0, \dots, 0)\right).$$

All this was done holding g fixed (so that $(y_o^1, \dots, y_o^n)$ was fixed), but now that we have our formula we can let g vary to get

$$\begin{aligned}\Psi \circ X(g) &= \Psi \circ X \circ \psi^{-1}(y^1, \dots, y^n)\\ &= \left(y^1, \dots, y^n, \sum_i a^i \frac{\partial \rho^1}{\partial x^i}(y^1, \dots, y^n, 0, \dots, 0), \dots, \sum_i a^i \frac{\partial \rho^n}{\partial x^i}(y^1, \dots, y^n, 0, \dots, 0)\right).\end{aligned}$$

Since the functions ρ^j are C^∞ on $\mathbb{R}^n \times \mathbb{R}^n$, this is a C^∞ function, so X is indeed a smooth vector field.

(c) Show that the tangent bundle of any Lie group is trivial.

Solution: Take a basis $\{f_1, \dots, f_n\}$ of T_eG. Use the process above to get smooth vector fields $X_1, \dots, X_n$ defined on G such that $X_j = (L_g)_*(f_j)$ for each j. Since each L_g is a diffeomorphism, we know $(L_g)_*$ is an isomorphism of tangent spaces, and thus if $\{f_1, \dots, f_n\}$ is a basis then so is $\{(L_g)_*(f_1), \dots, (L_g)_*(f_n)\}$. Hence the vector fields X_j are linearly independent at each point, and we can use Theorem 12.2.3 to conclude that TG is trivial.

3. Let H denote the Heisenberg group of matrices of the form

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \,\Big|\, x, y, z \in \mathbb{R} \right\},$$

with group operation given by matrix multiplication. As a manifold it is simply $\mathbb{R}^3$.

(a) Verify that this is a group and compute the left-translation maps in coordinates.

Solution: Write

$$A = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad X = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have

$$L_A(X) = AX = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+a & z+ay+c \\ 0 & 1 & y+b \\ 0 & 0 & 1 \end{pmatrix},$$

which is a matrix of the same form (so this is closed under multiplication). We verify that the inverse always exists: given (a, b, c) we can solve for (x, y, z) to get the identity via $x = -a$, $y = -b$, and $z = ab - c$.

Reading off the entries, in coordinates we have

$$\phi \circ L_A \circ \phi^{-1}(x, y, z) = (x + a, y + b, z + ay + c).$$

(b) Find a basis $\{v_1, v_2, v_3\}$ of $T_I G$ and compute the left-invariant vector fields generated by it.

Solution: First we compute the push-forward of the left-translation map for a fixed matrix A. Write $(u, v, w) = (\phi \circ L_A \circ \phi^{-1})(x, y, z) = (x+a, y+b, z+ay+c)$. Then using Proposition 11.2.1 gives

$$\begin{aligned}
(L_A)_* \left(\frac{\partial}{\partial x}\Big|_I \right) &= \frac{\partial u}{\partial x}\frac{\partial}{\partial u}\Big|_A + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\Big|_A + \frac{\partial w}{\partial x}\frac{\partial}{\partial w}\Big|_A \\
&= \frac{\partial}{\partial u}\Big|_A \\
(L_A)_* \left(\frac{\partial}{\partial y}\Big|_I \right) &= \frac{\partial}{\partial v}\Big|_A + a\frac{\partial}{\partial w}\Big|_A \\
(L_A)_* \left(\frac{\partial}{\partial z}\Big|_I \right) &= \frac{\partial}{\partial w}\Big|_A.
\end{aligned}$$

Therefore (renaming everything to (x, y, z)), a basis of left-invariant fields is

$$E_1 = \frac{\partial}{\partial x}\Big|_{(x,y,z)}, \qquad E_2 = \frac{\partial}{\partial y}\Big|_{(x,y,z)} + x\frac{\partial}{\partial z}\Big|_{(x,y,z)}, \qquad E_3 = \frac{\partial}{\partial z}\Big|_{(x,y,z)}.$$

4. Suppose $G\colon \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and 0 is a regular value of G. Let $M = G^{-1}\{0\}$ and let

$$N = \{(x, y, z, a, b, c) \mid G(x, y, z) = 0 \text{ and } aG_x(x, y, z) + bG_y(x, y, z) + cG_z(x, y, z) = 0\}.$$

(a) Show that N is a smooth submanifold of $\mathbb{R}^6$.

Solution: Obviously N is an inverse image of the function $H\colon \mathbb{R}^6 \to \mathbb{R}^2$ defined by

$$H(x, y, z, a, b, c) = \big(G(x, y, z), aG_x(x, y, z) + bG_y(x, y, z) + cG_z(x, y, z)\big);$$

then $N = H^{-1}\{(0,0)\}$. We need to check that $(0,0)$ is regular, i.e., that DH has maximal rank (two) everywhere on $H^{-1}\{(0,0)\}$. Of course we need to use the fact that G itself has maximal rank on $G^{-1}\{0\}$. We have

$$DH = \begin{pmatrix} G_x & G_y & G_z & 0 & 0 & 0 \\ aG_{xx} + bG_{xy} + cG_{xz} & aG_{xy} + bG_{yy} + cG_{yz} & aG_{xz} + bG_{yz} + cG_{zz} & G_x & G_y & G_z \end{pmatrix}.$$

To get the first row vector to be nonzero, we just need at least one of G_x, G_y, or G_z to be nonzero, which is true since DG has rank one on $G^{-1}\{0\}$. And to get the second row vector to be linearly independent of the first, we again just need at least one of G_x, G_y, or G_z to be nonzero. Note that the complicated part in the bottom left corner doesn't matter. Hence DH really does have rank two everywhere, and so $H^{-1}\{(0,0)\}$ is a smooth manifold.

(b) Show that N is a smooth vector bundle over M.

Solution: Define $\pi\colon N \to M$ as the restriction of the projection from $T\mathbb{R}^3$ to $\mathbb{R}^3$; that is, $\pi(x,y,z,a,b,c) = (x,y,z)$ whenever $(x,y,z,a,b,c) \in N$; then $(x,y,z) \in M$ for all such points, so $\pi[N]$ is indeed M. Since M and N are smooth submanifolds of $\mathbb{R}^3$ and $\mathbb{R}^6$, the map π is smooth. Given any $p \in M$ the inverse image is $\pi^{-1}\{p\} = \{p\} \times \{(a,b,c) \in \mathbb{R}^3 \,|\, aG_x + bG_y + cG_z = 0\}$, which is a two-dimensional vector space regardless of p.

To get the local trivializations, use the Implicit Function Theorem. At every point of M, at least one of G_x, G_y, or G_z is nonzero. Suppose $G_z(x_o, y_o, z_o) \neq 0$ at some $p = (x_o, y_o, z_o) \in M$; then there is an open set $U \subset \mathbb{R}^2$ with $(x_o, y_o) \in U$ and a smooth function $f\colon U \to \mathbb{R}$ such that $G(x,y,f(x,y)) = 0$ for all $(x,y) \in U$. The chart ϕ is then $(x,y,z) \mapsto (x,y)$, restricted to M. Tangent vectors are then given in the basis $\frac{\partial}{\partial x}\big|_{(x,y,f(x,y))}$ and $\frac{\partial}{\partial y}\big|_{(x,y,f(x,y))}$, and so the coordinate chart Φ induced from ϕ is just the restriction of $(x,y,z,a,b,c) \mapsto (x,y,a,b)$ to N. The fact that these maps are isomorphisms in each vector space comes from the fact that a vector (a,b,c) with $aG_x + bG_y + cG_z = 0$ is determined by a and b since we can solve for c (because $G_z \neq 0$). Similarly we'd get other charts when $G_x \neq 0$ or $G_y \neq 0$.

(c) Show that N is bundle-isomorphic to TM.

Solution: Let $\iota\colon M \to \mathbb{R}^3$; since M is a smooth submanifold, ι is smooth as a map of manifolds. Consider the induced bundle map $\iota_*\colon TM \to T\mathbb{R}^3$. For any curve $\gamma\colon \mathbb{R} \to M$ we have $\iota \circ \gamma\colon \mathbb{R} \to \mathbb{R}^3$ satisfying $G \circ \iota \circ \gamma(t) = 0$, and therefore $G_* \circ \iota_*(\gamma'(t)) = 0$. Thus if $\gamma(0) = p$ and $\gamma'(0) = v \in T_pM$, then $G(\iota(p)) = 0$ and $\iota_*(v)$ is in $\ker G_*$. Hence $\iota_*(v) \in N$. Furthermore the first three coordinates of $\iota_*(v)$ are a point (x,y,z) satisfying $G(x,y,z) = 0$, so that $\iota_*(v)$ lies over M. At any $p \in M$ the map $(\iota_*)_p$ is an isomorphism from T_pM to $T_{\iota(p)}\mathbb{R}^3$, and since ι_* preserves base points, it is a bundle isomorphism.