

Read “Introduction to Differential Geometry” through Chapter 12.

1. (Alternate approach to vectors.) For $p \in M$, let \mathcal{G}_p denote the set of all germs of C^∞ functions at p (that is, smooth real-valued functions defined on some open neighborhood of p under the equivalence relation that functions are equal if they coincide on some neighborhood of p which is contained in both their domains; see Remark 10.3.3). Define a *schmector* to be a linear operator D from the algebra \mathcal{G}_p to \mathbb{R} which satisfies the Leibniz rule $D(f \cdot g) = g(p)D(f) + f(p)D(g)$.

- (a) Prove that addition and multiplication are well-defined operations on germs (so that it is in fact an algebra), and that for any $v \in T_p M$ the operator $D_v: f \mapsto v(f)$ is a schmector.
- (b) Prove using the Leibniz rule that if $f: U \rightarrow \mathbb{R}$ is a constant function for some open set $U \ni p$, then $D(f) = 0$ for any schmector D .
- (c) Prove the following Lemma: for any C^∞ function $g: \mathbb{R}^n \rightarrow \mathbb{R}$, we can write

$$g(\mathbf{x}) = a + \sum_{k=1}^n b_k x^k + \sum_{i,j=1}^n c_{ij}(\mathbf{x}) x^i x^j,$$

where $a = g(\mathbf{0})$, $b_k = \frac{\partial g}{\partial x^k}(\mathbf{0})$, and $c_{ij}(\mathbf{x}) = \int_0^1 (1-t) \frac{\partial^2 g}{\partial x^i \partial x^j}(t\mathbf{x}) dt$. (Hint: Suppose for a fixed \mathbf{x} we denote $h(t) = g(tx^1, \dots, tx^n)$; show that $h''(t) = \sum_{i,j=1}^n x^i x^j \frac{\partial^2 g}{\partial x^i \partial x^j}(t\mathbf{x})$. Then use integration by parts.)

- (d) Use the previous result and the Leibniz rule to prove that the space of all schmectors is an n -dimensional vector space, and that the identification $v \mapsto D_v$ is an isomorphism from tangent vectors to schmectors.

2. A *Lie group* G is a smooth manifold such that

- G is a group under some multiplication operation,
 - the inversion $F: G \rightarrow G$ given by $F(g) = g^{-1}$ is a smooth map, and
 - the multiplication $P: G \times G \rightarrow G$ given by $P(g, h) = g \cdot h$ is a smooth map.
- (a) Show that for any fixed g the left-translation $L_g: G \rightarrow G$ given by $L_g(p) = g \cdot p$ is a diffeomorphism.
 - (b) If $e \in G$ is the identity and $v \in T_e G$ is any vector, show that $X_v(g) = (L_g)_*(v)$ defines a smooth vector field X_v on G . (Such a vector field is called “left-invariant.”)
 - (c) Show that the tangent bundle of any Lie group is trivial.

3. Let H denote the Heisenberg group of matrices of the form

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\},$$

with group operation given by matrix multiplication. As a manifold it is simply \mathbb{R}^3 .

- (a) Verify that this is a group and compute the left-translation maps in coordinates.
- (b) Find a basis $\{v_1, v_2, v_3\}$ of $T_I G$ and compute the left-invariant vector fields generated by it.

4. Suppose $G: \mathbb{R}^3 \rightarrow \mathbb{R}$ is a smooth function and 0 is a regular value of G . Let $M = G^{-1}\{0\}$ and let

$$N = \{(x, y, z, a, b, c) \mid G(x, y, z) = 0 \text{ and } aG_x(x, y, z) + bG_y(x, y, z) + cG_z(x, y, z) = 0\}.$$

- (a) Show that N is a smooth submanifold of \mathbb{R}^6 .
- (b) Show that N is a smooth vector bundle over M .
- (c) Show that N is bundle-isomorphic to TM .