Math 70900 Homework #5 Solutions

1. If $n \in \mathbb{N}$ and $\Omega = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$, let $Sp_{2n}(\mathbb{R}) = \{P \in \mathbb{R}^{2n\times 2n} : P^T\Omega P = \Omega\}$ be the symplectic group.

   (a) If $P = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ for $n\times n$ matrices A, B, C, D, work out the condition for P to be symplectic explicitly.

   Solution: We have

$$\begin{aligned}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} &= \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}\begin{pmatrix} C & D \\ -A & -B \end{pmatrix} \\ &= \begin{pmatrix} A^TC - C^TA & A^TD - C^TB \\ B^TC - D^TA & B^TD - D^TB \end{pmatrix}\end{aligned}$$

   Thus we must have $A^TC = C^TA$ and $B^TD = D^TB$, together with $A^TD - C^TB = I$. The fourth condition on $B^TC - D^TA = -I$ is simply the negative transpose of the other condition we have.

   (b) Define $F\colon \mathbb{R}^{2n\times 2n} \to \mathbb{R}^{n(2n-1)}$ by $F(P) = P^T\Omega P$. Compute $DF(P)(Q)$ as in class at any matrix $Q = \left(\begin{smallmatrix} W & X \\ Y & Z \end{smallmatrix}\right)$.

   Solution: The derivative of the defining condition is

$$\begin{aligned} DF(P)(Q) &= Q^T\Omega P + P^T\Omega Q \\ &= \begin{pmatrix} W^T & Y^T \\ X^T & Z^T \end{pmatrix}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\begin{pmatrix} W & X \\ Y & Z \end{pmatrix} \\ &= \begin{pmatrix} W^T & Y^T \\ X^T & Z^T \end{pmatrix}\begin{pmatrix} C & D \\ -A & -B \end{pmatrix} + \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}\begin{pmatrix} Y & Z \\ -W & -X \end{pmatrix} \\ &= \begin{pmatrix} W^TC - Y^TA & W^TD - Y^TB \\ X^TC - Z^TA & X^TD - Z^TB \end{pmatrix} + \begin{pmatrix} A^TY - C^TW & A^TZ - C^TX \\ B^TY - D^TW & B^TZ - D^TX \end{pmatrix}. \end{aligned}$$

   (c) Show that $DF(P)$ is surjective onto the antisymmetric $2n \times 2n$ matrices for every symplectic P, and conclude that $Sp_{2n}(\mathbb{R})$ is a smooth manifold.

   Solution: Let's apply the same trick that worked in class for the orthogonal group. That is, we compute

$$DF(P)(PQ) = (PQ)^T\Omega P + P^T\Omega(PQ) = Q^TP^T\Omega P + P^T\Omega PQ = Q^T\Omega + \Omega Q.$$

   We want to solve $DF(P)(Q) = \left(\begin{smallmatrix} E & F \\ -F^T & G \end{smallmatrix}\right)$ for $Q = \left(\begin{smallmatrix} W & X \\ Y & Z \end{smallmatrix}\right)$, where E and G are antisymmetric. We have

$$DF(P)(PQ) = \begin{pmatrix} Y - Y^T & W^T + Z \\ -Z^T - W & X^T - X \end{pmatrix} = \begin{pmatrix} E & F \\ -F^T & G \end{pmatrix},$$

   where one solution is

$$Y = \tfrac{1}{2}E, \qquad W = \tfrac{1}{2}F^T, \qquad X = \tfrac{1}{2}G, \qquad Z = -\tfrac{1}{2}F.$$

   Hence $DF(P)$ is surjective for every $P \in SP_{2n}(\mathbb{R})$, and thus $SP_{2n}(\mathbb{R})$ is a smooth manifold.

2. Consider the Grassmannian manifold Gr(2, 4). Let $\{e_1, e_2, e_3, e_4\}$ denote the standard basis of $\mathbb{R}^4$. Let (ϕ, U) denote the coordinate chart generated as in the text by vectors $\{e_1, e_2\}$ and (ψ, V) denote the chart generated by vectors $\{e_1, e_3\}$. Compute the transition map explicitly on the overlap.

   Solution: Consider a plane spanned by the two vectors $\{(1, 0, p, q), (0, 1, r, s)\}$, and also spanned by the two vectors $\{(1, w, 0, x), (0, y, 1, z)\}$. We want to know under what conditions these spans are equal, and it is sufficient to be able to express each vector of one basis in terms of the others. That is, we want to solve the equations

   $$a(1, 0, p, q) + b(0, 1, r, s) = (1, w, 0, x)$$
   $$c(1, 0, p, q) + d(0, 1, r, s) = (0, y, 1, z)$$

   for $a, b, c, d \in \mathbb{R}$.

   These equations simplify to

   $$a = 1, b = w, ap + br = 0, aq + bs = x$$
   $$c = 0, d = y, cp + dr = 1, cq + ds = z$$

   We must have $a = 1$, $b = w$, $c = 0$, and $d = y$, and now our goal is to solve for (p, q, r, s): we get

   $$p = -\frac{w}{y}, \qquad q = x - \frac{w}{z}, \qquad r = \frac{1}{y}, \qquad s = \frac{1}{z}.$$

   This is defined on the open set $y \neq 0$ and $z \neq 0$ and is obviously C^∞.

3. Verify that for any $k < n$, the map defining the Stiefel manifold $F(A) = A^T A$ from $n \times k$ matrices A to symmetric $k \times k$ matrices has I_k as a regular value. (Hint: this works the same way as for the orthogonal group.)

   Solution: The derivative of F is $DF(A)(B) = A^T B + B^T A$. We want to prove that it is surjective for any A satisfying $A^T A = I_k$. Let C be any symmetric $k \times k$ matrix, and define $B = \frac{1}{2} AC$, which is an $n \times k$ matrix. Then we check that

   $$DF(A)(B) = \tfrac{1}{2} A^T AC + \tfrac{1}{2} C^T A^T A = \tfrac{1}{2}(C + C^T) = C,$$

   as desired.

4. If $F \colon \mathbb{R}^{n+1} \to \mathbb{R}$ has a regular value $r_0 \in \mathbb{R}$, then $M = F^{-1}[r_0]$ is a smooth submanifold. Show that it must be orientable.

   Solution: For each point of M there is a coordinate chart (φ, U) on $\mathbb{R}^n$ such that $M \cap U = \varphi^{-1}\{(\mathbf{x},) : \mathbf{x} \in \mathbb{R}^n\}$. In these coordinates the function F can be expressed as $F(x^1, \ldots, x^n, x^{n+1})$, and since F is constant on M, we must have $\frac{\partial F}{\partial x^i}(\mathbf{x}, 0)$ for every $\mathbf{x} \in \mathbb{R}^n$. Since F has rank one everywhere on M, we must have $\frac{\partial F}{\partial x^{n+1}}(\mathbf{x}, 0)$ nowhere zero. We may then replace the coordinate chart with $(y^1, \ldots, y^n, y^{n+1})$ where $y^i = x^i$ for $i \leq n$ and $y^{n+1} = F(x^1, \ldots, x^n)$, and this is a genuine chart (a diffeomorphism of $\mathbb{R}^{n+1}$ since $\frac{\partial F}{\partial x^{n+1}}$ is nonzero.

The coordinates $(x^1, \ldots, x^n, x^{n+1})$ are either compatible with the standard orientation on $\mathbb{R}^{n+1}$ or not; if they are not compatible we reflect in one of the other coordinates (such as x^n) in order to get a compatible orientation. In this way we cover all of M with coordinate charts on $\mathbb{R}^{n+1}$ that are compatible with the known orientation on $\mathbb{R}^{n+1}$. Now we restrict these to the actual coordinate charts on M by taking just the first n components of each of these special charts.

When transitioning between two such charts $(y^1, \ldots, y^n, y^{n+1})$ and $(z^1, \ldots, z^n, z^{n+1})$, we have positive determinant of the Jacobian, and since $z^{n+1} = y^{n+1}$, the Jacobian matrix will look like $\left(\begin{smallmatrix} Z & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in block form. Thus the determinant of Z is also positive. Now consider the restriction to $y^{n+1} = 0$ to get coordinate charts $(y^1, \ldots, y^n)$ on M that cover all of M; the transition maps will also have positive determinant and satisfy the orientability condition.

5. Recall that the stereographic projection transition map on S^2 is given by $(x, y) = \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)$. Use this to express the vector given in north-pole coordinates by $\frac{\partial}{\partial v}\big|_p$, in terms of the south-pole coordinate vectors $\frac{\partial}{\partial x}\big|_p$ and $\frac{\partial}{\partial y}\big|_p$, for a point p on the sphere which isn't one of the poles. What happens to $\frac{\partial}{\partial v}\big|_p$ as p approaches the south pole?

   Solution: Using the transition formula, we have

$$\begin{aligned} \frac{\partial}{\partial v}\bigg|_p &= \frac{\partial x}{\partial v}\bigg|_{(u(p),v(p)}\frac{\partial}{\partial x}\bigg|_p + \frac{\partial y}{\partial v}\bigg|_{(u(p),v(p)}\frac{\partial}{\partial y}\bigg|_p \\ &= -\frac{2uv}{(u^2+v^2)^2}\frac{\partial}{\partial x}\bigg|_p + \frac{u^2-v^2}{(u^2+v^2)^2}\frac{\partial}{\partial y}\bigg|_p. \end{aligned}$$

   Now plugging in our formulas for u and v in terms of x and y, this becomes

$$\frac{\partial}{\partial v}\bigg|_p = -2xy\,\frac{\partial}{\partial x}\bigg|_p + (x^2 - y^2)\,\frac{\partial}{\partial y}\bigg|_p.$$

   As we approach the south pole, x and y both approach 0 so the vector $\frac{\partial}{\partial v}\big|_p$ approaches the zero vector as p approaches the south pole.

6. Consider the function $f\colon \mathbb{C} \to \mathbb{R}$ defined by $f(z) = \operatorname{Im}(z^3)$, and let $\gamma(t) = e^{it}$.

   (a) Compute $(f \circ \gamma)'(0)$ directly.

   Solution: We have $f \circ \gamma(t) = \operatorname{Im}(e^{3it}) = \sin 3t$, so that $(f \circ \gamma)'(t) = 3\cos 3t$ and $(f \circ \gamma)'(0) = 3$.

   (b) Compute $f \circ \mathbf{x}^{-1}$, $\mathbf{x} \circ \gamma$, and $(f \circ \gamma)'(0)$ by the Chain Rule, using rectangular coordinates $\mathbf{x} = (x, y)$.

   Solution: In rectangular coordinates, we have $f \circ \mathbf{x}^{-1}(x, y) = 3x^2y - y^3$, and $\mathbf{x} \circ \gamma(t) = (\cos t, \sin t)$. Thus we have at time zero that

$$\frac{d}{dt}(x \circ \gamma(t))\bigg|_{t=0} = -\sin t\big|_{t=0} = 0, \qquad \frac{d}{dt}(y \circ \gamma(t))\bigg|_{t=0} = \cos t\big|_{t=0} = 1.$$

Meanwhile since $\gamma(0) = (1,0)$ in (x,y) coordinates, we have

$$\frac{\partial}{\partial x}(f \circ \mathbf{x}^{-1})\Big|_{(1,0)} = 6xy\Big|_{(1,0)} = 0, \qquad \frac{\partial}{\partial y}(f \circ \mathbf{x}^{-1})\Big|_{(1,0)} = 3x^2 - 3y^2\Big|_{(1,0)} = 3.$$

Thus we get

$$\begin{aligned}\frac{d}{dt}(f \circ \gamma)(t)\Big|_{t=0} &= \frac{\partial(f \circ \mathbf{x}^{-1})}{\partial x}\Big|_{(1,0)} \frac{d(x \circ \gamma)}{dt}\Big|_{t=0} + \frac{\partial(f \circ \mathbf{x}^{-1})}{\partial y}\Big|_{(1,0)} \frac{d(y \circ \gamma)}{dt}\Big|_{t=0} \\ &= 0 \cdot 0 + 3 \cdot 1 = 3.\end{aligned}$$

(c) Compute $f \circ \mathbf{u}^{-1}$, $\mathbf{u} \circ \gamma$, and $(f \circ \gamma)'(0)$ by the Chain Rule, using polar coordinates $\mathbf{u} = (r, \theta)$.

Solution: In polar coordinates the curve is described by $r = 1$, $\theta = t$, while the function is described by $f \circ \mathbf{u}^{-1}(r,\theta) = \mathrm{Im}(r^3 e^{3i\theta}) = r^3 \sin 3\theta$. Thus the derivative is given by

$$\frac{d}{dt}(f \circ \gamma)(t)\Big|_{t=0} = 3r^2 \sin 3\theta\Big|_{(1,0)} \cdot 0 + 3r^3 \cos 3\theta\Big|_{(1,0)} \cdot 1 = 3.$$