Math 70900 Homework #5
due **Friday**, October 6

Read "Introduction to Differential Geometry" through Chapter 10.

1. If $n \in \mathbb{N}$ and $\Omega = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$, let $Sp_{2n}(\mathbb{R}) = \{P \in \mathbb{R}^{2n\times 2n} : P^T \Omega P = \Omega\}$ be the symplectic group.

   (a) If $P = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ for $n \times n$ matrices A, B, C, D, work out the condition for P to be symplectic explicitly.

   (b) Define $F\colon \mathbb{R}^{2n\times 2n} \to \mathbb{R}^{n(2n-1)}$ by $F(P) = P^T \Omega P$. Compute $DF(P)(Q)$ as in class at any matrix $Q = \left(\begin{smallmatrix} W & X \\ Y & Z \end{smallmatrix}\right)$.

   (c) Show that $DF(P)$ is surjective onto the antisymmetric $2n \times 2n$ matrices for every symplectic P, and conclude that $Sp_{2n}(\mathbb{R})$ is a smooth manifold.

2. Consider the Grassmannian manifold $\mathrm{Gr}(2, 4)$. Let $\{e_1, e_2, e_3, e_4\}$ denote the standard basis of $\mathbb{R}^4$. Let (ϕ, U) denote the coordinate chart generated as in the text by vectors $\{e_1, e_2\}$ and (ψ, V) denote the chart generated by vectors $\{e_1, e_3\}$. Compute the transition map explicitly on the overlap.

3. Verify that for any $k < n$, the map defining the Stiefel manifold $F(A) = A^T A$ from $n \times k$ matrices A to symmetric $k \times k$ matrices has I_k as a regular value. (Hint: this works the same way as for the orthogonal group.)

4. If $F\colon \mathbb{R}^{n+1} \to \mathbb{R}$ has a regular value $r_0 \in \mathbb{R}$, then $M = F^{-1}[r_0]$ is a smooth submanifold. Show that it must be orientable.

5. Recall that the stereographic projection transition map on S^2 is given by $(x, y) = \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)$. Use this to express the vector given in north-pole coordinates by $\frac{\partial}{\partial v}\big|_p$, in terms of the south-pole coordinate vectors $\frac{\partial}{\partial x}\big|_p$ and $\frac{\partial}{\partial y}\big|_p$, for a point p on the sphere which isn't one of the poles. What happens to $\frac{\partial}{\partial v}\big|_p$ as p approaches the south pole?

6. Consider the function $f\colon \mathbb{C} \to \mathbb{R}$ defined by $f(z) = \mathrm{Im}(z^3)$, and let $\gamma(t) = e^{it}$.

   (a) Compute $(f \circ \gamma)'(0)$ directly.

   (b) Compute $f \circ \mathbf{x}^{-1}$, $\mathbf{x} \circ \gamma$, and $(f \circ \gamma)'(0)$ by the Chain Rule, using rectangular coordinates $\mathbf{x} = (x, y)$.

   (c) Compute $f \circ \mathbf{u}^{-1}$, $\mathbf{u} \circ \gamma$, and $(f \circ \gamma)'(0)$ by the Chain Rule, using polar coordinates $\mathbf{u} = (r, \theta)$.