Math 70900 Homework #4 Solutions

1. The torus in $\mathbb{R}^3$ can be defined to be the image $\mathbb{T}^2 = F[\mathbb{R}^2]$ of $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$F(u,v) = \big((2+\cos u)\cos v, (2+\cos u)\sin v, \sin u\big),$$

or as the inverse image $H^{-1}\{1\}$ where $H\colon \mathbb{R}^3 \to \mathbb{R}$ is given by

$$H(x,y,z) = \big(\sqrt{x^2+y^2} - 2\big)^2 + z^2.$$

(a) Show that $F[\mathbb{R}^2] = H^{-1}\{1\}$ as sets in $\mathbb{R}^3$.
Solution: If $(x,y,z) = \big((2+\cos u)\cos v, (2+\cos u)\sin v, \sin u\big)$, then we have $x^2+y^2 = (2+\cos u)^2$, and since $2+\cos u$ is positive, we can write $\sqrt{x^2+y^2} = 2+\cos u$. Thus $H(x,y,z) = \cos^2 u + \sin^2 u = 1$.
Conversely, if $H(x,y,z) = 1$, then $\sqrt{x^2+y^2} - 2 = \cos u$ and $z = \sin u$ for some $u \in \mathbb{R}$. Hence $\sqrt{x^2+y^2} = 2+\cos u$, and we conclude that $x = (2+\cos u)\cos v$ and $y = (2+\cos u)\sin v$ for some $v \in \mathbb{R}$. Thus $(x,y,z) = F(u,v)$ for some $(u,v) \in \mathbb{R}^2$.

(b) Show that DF has maximal rank everywhere.
Solution: The matrix DF is given by

$$DF = \begin{pmatrix} -\sin u\cos v & -(2+\cos u)\sin v \\ -\sin u \sin v & (2+\cos u)\cos v \\ \cos u & 0 \end{pmatrix}.$$

We compute the subdeterminants: letting D_{ij} denote the determinants using row i and j, we get

$$D_{12} = -(2+\cos u)\sin u, \quad D_{13} = \cos u(2+\cos u)\sin v, \quad D_{23} = -\cos u(2+\cos u)\cos v.$$

An easy way to see that at least one of these is nonzero is to compute

$$D_{12}^2 + D_{13}^2 + D_{23}^2 = (2+\cos u)^2,$$

which is never zero.

(c) Show that DH has maximal rank everywhere on $H^{-1}\{1\}$.
Solution: The matrix DH is given by

$$DH = \begin{pmatrix} \frac{2x(\sqrt{x^2+y^2}-2)}{\sqrt{x^2+y^2}} & \frac{2y(\sqrt{x^2+y^2}-2)}{\sqrt{x^2+y^2}} & 2z \end{pmatrix}.$$

Doing the same trick as before, we get

$$H_x^2 + H_y^2 + H_z^2 = 4(\sqrt{x^2+y^2}-2)^2 + 4z^2 = 4H(x,y,z) = 4$$

for all points in $H^{-1}\{1\}$. So again at least one of these must be nonzero, and DH always has rank 1.

2. Suppose $U \subset \mathbb{R}^n$ is open and that $\phi\colon U \to \mathbb{R}^n$ is a C^∞ map. Furthermore suppose that ϕ is injective and that $D\phi(x)$ is an invertible matrix at every $x \in U$. Prove that $V = \phi[U]$ is open in $\mathbb{R}^n$ and that ϕ is a diffeomorphism from U to V. (That is, ϕ is a homeomorphism and ϕ^{-1} is C^∞.)

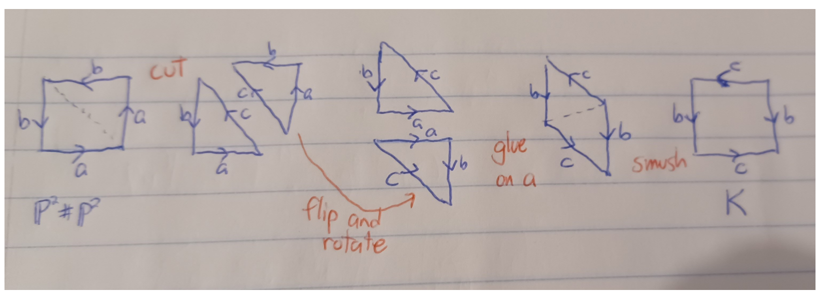
   Solution: Let $y_o \in V = \phi[U]$; then $y_o = \phi(x_o)$ for a unique $x_o \in U$, and we know $D\phi(x_o)$ is invertible. Hence by the inverse function theorem there is an open set $W \ni y_o$ and a smooth function $G\colon W \to U$ such that $\phi(G(y)) = y$ for all $y \in W$. The only way this makes sense is if $G(y) \in U$ for all $y \in W$, which means W is in the image of ϕ. Since every y_o in $\phi[U]$ is contained in an open set W in $\phi[U]$, we conclude that $\phi[U]$ is open.

   Since continuity and C^∞ of ϕ^{-1} are local properties, we just need to check them in an open neighborhood of each $y_o \in V$. But we just did this: the function G defined on a neighborhood of y_o is smooth, and by uniqueness of inverses must agree with ϕ^{-1} on that neighborhood.

3. Let $\mathbb{K}$ denote the Klein bottle, defined by the polygon identification in Figure 8.18.

   (a) By cutting and pasting polygons, prove that $\mathbb{P}^2 \# \mathbb{P}^2 \cong \mathbb{K}$.

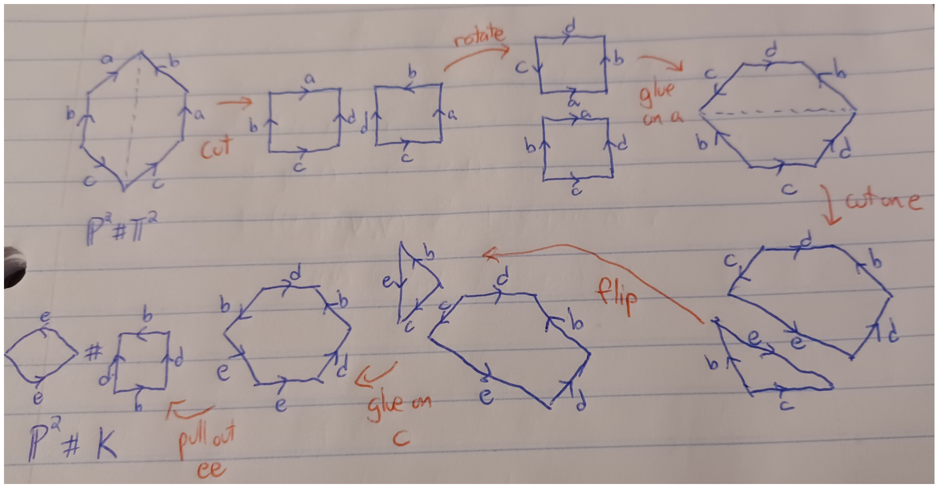
   Solution: There is pretty much only one way to do this; write $\mathbb{P}^2 \# \mathbb{P}^2$ as a square $aabb$, then split along the diagonal to get two triangles abc^{-1} and bac. Join these along side a to obtain the word $cb^{-1}cb$, which is the usual Klein bottle.



   (b) By cutting and pasting polygons, prove that $\mathbb{P}^2 \# \mathbb{K} \cong \mathbb{P}^2 \# \mathbb{T}^2$, so that $\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \cong \mathbb{P}^2 \# \mathbb{T}^2$.

   Solution: I start with $\mathbb{P}^2 \# \mathbb{T}^2$, written as $aba^{-1}b^{-1}cc$. Then I draw a line between the two c sides and send it to the opposite side of the hexagon (where it falls between a and b). I get two squares with words $ad^{-1}c^{-1}b$ and $a^{-1}c^{-1}db^{-1}$, which I join along the side a to obtain the hexagon with word $db^{-1}d^{-1}c^{-1}bc^{-1}$.

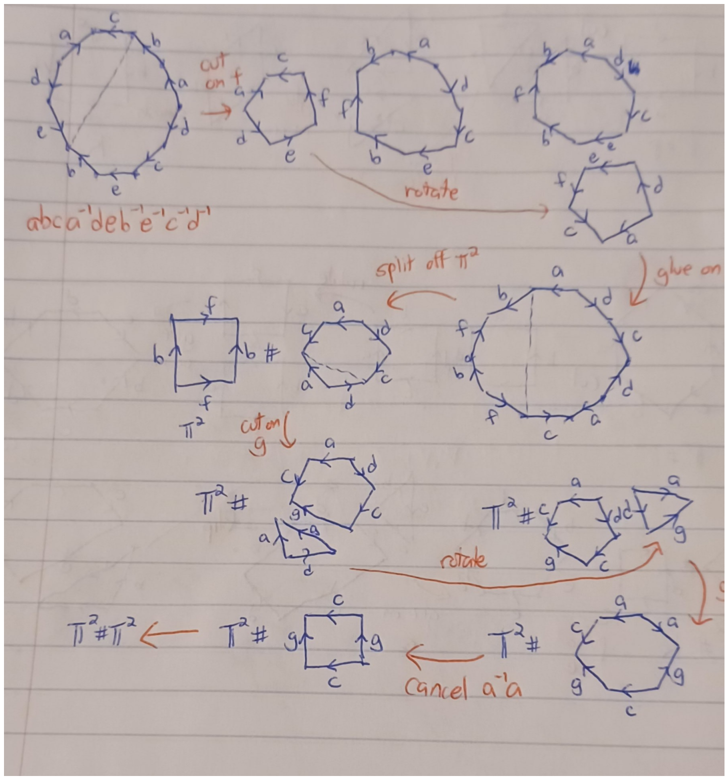
   Now I use the Third Reduction, cutting from one head of c to the other head of c. I have a pentagon $edbd^{-1}c$ and a triangle $ce^{-1}b^{-1}$, which I join along edge c to get hexagon $b^{-1}db^{-1}d^{-1}e^{-1}e^{-1}$. Pulling out the projective plane $e^{-1}e^{-1}$, I obtain $b^{-1}db^{-1}d^{-1}$, which is a Klein bottle. So I have $\mathbb{P}^2 \# \mathbb{K}$.



4. Consider a surface generated by a decagon with sides identified according to the word $abca^{-1}deb^{-1}e^{-1}c^{-1}d^{-1}$. By using the reductions from Lemmas 8.2.6–8.2.9, reduce the surface to a connected sum of tori and projective planes, and determine the genus and orientability. (To avoid having to draw too many decagons, remember to pull out projective planes or tori as soon as you've found them.)

   Solution: There are no projective planes in this manifold, since there are no sides that go in the same direction twice. Thus I have to use the Fourth Reduction. Since there is a combination $eb^{-1}e^{-1}$, I'm halfway done with a Fourth Reduction already. So I join the head of one b to the head of another b with an arrow f. Splitting it up I have a septagon $abf^{-1}b^{-1}e^{-1}c^{-1}d^{-1}$ and a pentagon $efca^{-1}d$, which I join along edge e.

   I obtain the decagon $abf^{-1}b^{-1}fca^{-1}dc^{-1}d^{-1}$. Pulling out the component $bf^{-1}b^{-1}f$ (which is a torus), I obtain the connected sum of $\mathbb{T}^2$ with the hexagon $aca^{-1}dc^{-1}d^{-1}$. Again I notice the term aca^{-1} which means I'm halfway done with a Fourth Reduction. So I draw an arrow g from one head of c to another head of c, obtaining a pentagon $acg^{-1}c^{-1}d^{-1}$ and a triangle dga^{-1}. Gluing these along edge d, I obtain hexagon $acg^{-1}c^{-1}ga^{-1}$. The edges a cancel out by the First Reduction, and I'm left with $cg^{-1}c^{-1}g$, which is another torus.



5. For $a \in \mathbb{R}$, let M_a be the set of 2×2 real matrices with trace 2 and determinant a. Show that M_a is a manifold if and only if $a \neq 1$. (Hint for $a = 1$: show that the defining condition looks like the equation of a cone in $\mathbb{R}^3$.)

   Solution: Identify 2×2 real matrices with $\mathbb{R}^4$, and define a function $F\colon \mathbb{R}^4 \to \mathbb{R}^2$ by $F(A) = (\operatorname{Tr} A, \det A)$. We then want to examine the inverse image of $(2, a)$.

   In coordinates where the matrix is $A = \left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right)$, we have

$$F(w, x, y, z) = (w + z, wz - xy).$$

   Thus we have

$$DF(w, x, y, z) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ z & -y & -x & w \end{pmatrix}.$$

   If either x or y is nonzero, this is obviously rank two. If both x and y are zero, this is rank two unless $w = z$. In that case it reduces to rank one.

   So with the constraints $w = z$ and $w + z = 2$ and $x = y = 0$ and $wz - xy = a$, we must obviously have $w = z = 1$ and thus $a = 1$ as well to get rank one. So if $a \neq 1$ then DF has rank two, and the implicit function theorem tells us it's a manifold.

   To see what happens when $a = 1$, we express the matrix in a slightly different form. Write $A = \left(\begin{smallmatrix} 1+p & q-r \\ q+r & 1-p \end{smallmatrix}\right)$, which can always be done when A has trace two. Then

$$\det A = 1 - p^2 - (q^2 - r^2) = 1$$

   implies that

$$r^2 = p^2 + q^2,$$

which is exactly the equation of a cone in 3-space, and clearly not even a topological manifold (though a rigorous proof is not necessarily easy).