Math 70900 Homework #4
due Wednesday, September 27

Read "Introduction to Differential Geometry" through Chapter 8.

1. The torus in $\mathbb{R}^3$ can be defined to be the image $\mathbb{T}^2 = F[\mathbb{R}^2]$ of $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$F(u,v) = \big((2+\cos u)\cos v, (2+\cos u)\sin v, \sin u\big),$$

or as the inverse image $H^{-1}\{1\}$ where $H\colon \mathbb{R}^3 \to \mathbb{R}$ is given by

$$H(x,y,z) = \big(\sqrt{x^2+y^2} - 2\big)^2 + z^2.$$

   (a) Show that $F[\mathbb{R}^2] = H^{-1}\{1\}$ as sets in $\mathbb{R}^3$.

   (b) Show that DF has maximal rank everywhere.

   (c) Show that DH has maximal rank everywhere on $H^{-1}\{1\}$.

2. Suppose $U \subset \mathbb{R}^n$ is open and that $\phi\colon U \to \mathbb{R}^n$ is a C^∞ map. Furthermore suppose that ϕ is injective and that $D\phi(x)$ is an invertible matrix at every $x \in U$. Prove that $V = \phi[U]$ is open in $\mathbb{R}^n$ and that ϕ is a diffeomorphism from U to V. (That is, ϕ is a homeomorphism and ϕ^{-1} is C^∞.)

3. Let $\mathbb{K}$ denote the Klein bottle, defined by the polygon identification in Figure 8.18.

   (a) By cutting and pasting polygons, prove that $\mathbb{P}^2 \# \mathbb{P}^2 \cong \mathbb{K}$.

   (b) By cutting and pasting polygons, prove that $\mathbb{P}^2 \# \mathbb{K} \cong \mathbb{P}^2 \# \mathbb{T}^2$, so that $\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \cong \mathbb{P}^2 \# \mathbb{T}^2$.

4. Consider a surface generated by a decagon with sides identified according to the word $abca^{-1}deb^{-1}e^{-1}c^{-1}d^{-1}$. By using the reductions from Lemmas 8.2.6–8.2.9, reduce the surface to a connected sum of tori and projective planes, and determine the genus and orientability. (To avoid having to draw too many decagons, remember to pull out projective planes or tori as soon as you've found them.)

5. For $a \in \mathbb{R}$, let M_a be the set of 2×2 real matrices with trace 2 and determinant a. Show that M_a is a manifold if and only if $a \neq 1$. (Hint for $a = 1$: show that the defining condition looks like the equation of a cone in $\mathbb{R}^3$.)