Math 70900 Homework #3 Solutions

1. Consider the function $F\colon \mathbb{R}^3 \to \mathbb{R}$ given by

$$F(x, y, z) = x^2 + y^2 + z^2 + 2xyz,$$

and the level surfaces S_c consisting of the points satisfying $F(x, y, z) = c$ for $c > 0$.

(a) Show using the Implicit Function Theorem that if $c \neq 1$, then we can always locally represent S_c as the graph of a smooth function of one of the variables in terms of the others.

Solution: We compute DF first and get

$$DF(x, y, z) = \begin{pmatrix} F_x & F_y & F_z \end{pmatrix} = \begin{pmatrix} 2x + 2yz & 2y + 2xz & 2z + 2xy \end{pmatrix}.$$

The only way this does not have rank one is if it's identically zero, and the only way that can happen is if there is a point $(x, y, z) \in S_c$ such that

$$x + yz = 0, \qquad y + xz = 0, \qquad z + xy = 0.$$

Eliminating z using $z = -xy$, we get

$$x(1 - y^2) = 0, \qquad y(1 - x^2) = 0.$$

So either $x = y = 0$ or $|x| = |y| = 1$. The equations $x = y = 0$ imply that $z = 0$, so that $c = 0$, but we assumed $c > 0$. So suppose $|x| = |y| = 1$. If x and y have the same sign, then z has the opposite; if x and y have opposite signs then z is the same as one of them. Thus we must have two signs the same and one sign different. If two of them are negative and one is positive (for example x, y negative) then $z + xy = 2$ and we do not get a critical point. So the only points to worry about are $x = 1$, $y = 1$, $z = -1$ and permutations of those, which all give $F(1, 1, -1) = 1$. We conclude $c = 1$ is the only point where the Implicit Function Theorem does not apply.

(b) What happens when $c = 1$? (Try plotting it.)

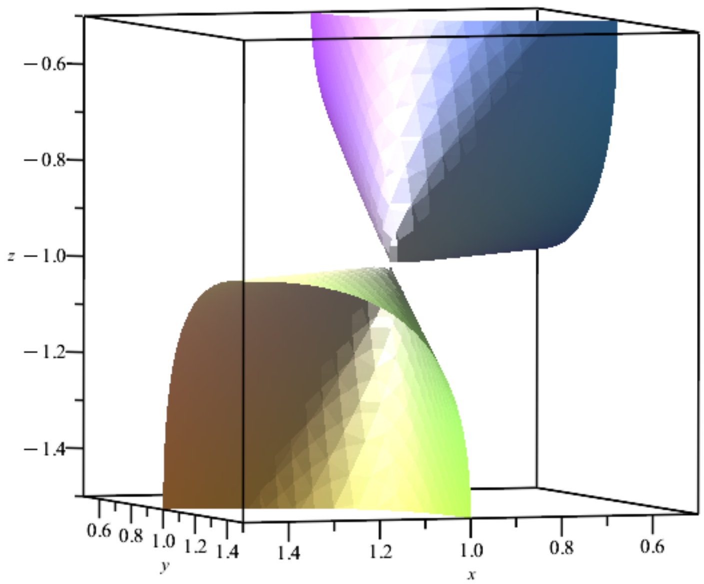
Solution: When $c = 1$ we have the equation

$$x^2 + y^2 + z^2 + 2xyz = 1,$$

and from above we know this is a manifold everywhere except at points like $(1, 1, -1)$ and permutations of it. Solving for z near $(1, 1, -1)$, we would get

$$z = -xy \pm \sqrt{(x + 1)(y + 1)(x - 1)(y - 1)},$$

and we see that z can be solved in the square $x \geq 1$,$y \geq 1$ and the square $x \leq 1$, $y \leq 1$ with a cusp joining these two squares. Hence the singularity looks roughly like a cone, as pictured.



2. For the differential equation

$$\frac{dx}{dt} = x^2, \qquad x(0) = 1,$$

use Picard iteration to obtain the approximate solutions up to $k = 2$. Check that $x(t) = \frac{1}{1-t}$ is the exact solution and compare your functions $\eta_k(t)$ to its Taylor series.

Solution: The Picard iteration algorithm in this case is

$$\eta_{k+1}(t) = 1 + \int_0^t \eta_k(s)^2\, ds, \qquad \eta_0(t) = 1.$$

We compute

$$\eta_1(t) = 1 + \int_0^t 1\, ds = 1 + t$$

and thus

$$\eta_2(t) = 1 + \int_0^t (1+s)^2\, ds = 1 + t + t^2 + \tfrac{1}{3}t^3.$$

To check the exact solution, we just plug in $t = 0$ to get $x(0) = 1$ as desired, and check the derivative $x'(t) = (1-t)^{-2} = x(t)^2$ as desired.

The series for $x(t)$ is

$$x(t) = \sum_{k=0}^{\infty} t^k,$$

which matches $\eta_2(t)$ in the first three terms.

3. Consider the coordinate chart $(x, y) = F(u, v) = (v \cos u, \sin u/v)$. Find the largest open set U around $(u, v) = (0, 1)$ such that F is a diffeomorphism on U (i.e., F is smooth, invertible, and F^{-1} is also smooth). What is the image of U in the plane? What do the coordinate curves look like?

   Solution: I meant for this to be interpreted as $F(u, v) = (v \cos u, v^{-1} \sin u)$, and I apologize to everyone who interpreted it as $\sin(u/v)$.

   The Jacobian determinant is

$$\begin{aligned}\text{Jac}(u, v) = x_u y_v - x_v y_u &= (-v \sin u)(-v^{-2} \sin u) - (\cos u)(v^{-1} \cos u \\ &== v^{-1}(\sin^2 u - \cos^2 u) = -v^{-1} \cos(2u).\end{aligned}$$

   So we need $v > 0$ and for $-\frac{\pi}{4} < u < \frac{\pi}{4}$ to have the largest open set containing $(0, 1)$.

   For fixed u, the coordinate curves satisfy $xy = \sin u \cos u = \frac{1}{2} \sin 2u = \text{const}$, so they are hyperbolas. If $0 < u < \frac{\pi}{4}$ then x and y are both positive and we traverse the hyperbolas $xy = C$ for $0 < C < \frac{1}{2}$ in the first quadrant. If $-\frac{\pi}{4} < u < 0$ then x is positive while y is negative, and we traverse the hyperbolas $xy = -C$ for $0 < C < \frac{1}{2}$ in the fourth quadrant.

   For fixed v, the coordinate curves satisfy

$$\frac{x^2}{v^2} + v^2 y^2 = 1,$$

   which are ellipses through $(v, 0)$ and $(0, v^{-1})$, traversed only from $-\frac{\pi}{4} < u < \frac{\pi}{4}$.

   The image of coordinate curves under the map is shown below.

4. Suppose M is a set, I is some index set, and we have a collection of sets $U_\alpha \subset M$ and bijective functions ϕ_α which map U_α onto $\mathbb{R}^n$ for each $\alpha \in I$. Suppose that the union of all U_α is M. Define a set $\Omega \subset M$ to be open if and only if $\phi_\alpha[\Omega \cap U_\alpha]$ is open for every $\alpha \in I$. Check that this definition satisfies the conditions for a topology on M.

   If we further demand that whenever $U_\alpha \cap U_\beta$ is nonempty, the set $\phi_\alpha[U_\alpha \cap U_\beta]$ is open in $\mathbb{R}^n$ and the function $\phi_\alpha \circ \phi_\beta^{-1} \colon \phi_\beta[U_\alpha \cap U_\beta] \subset \mathbb{R}^n \to \phi_\alpha[U_\alpha \cap U_\beta] \subset \mathbb{R}^n$ is a homeomorphism, show that each U_α is open in this topology, and that each ϕ_α is continuous in this topology.

   Solution: The empty set is trivial, and the entire set $\Omega = M$ is open since $\phi_\alpha[U_\alpha] = \mathbb{R}^n$ for each α. If Ω_1 and Ω_2 are open, then for any α,

$$\phi_\alpha[\Omega_1 \cap \Omega_2 \cap U_\alpha] = \phi_\alpha\big[(\Omega_1 \cap U_\alpha) \cap (\Omega_2 \cap U_\alpha)\big] = \phi_\alpha[\Omega_1 \cap U_\alpha] \cap \phi_\alpha[\Omega_2 \cap U_\alpha]$$

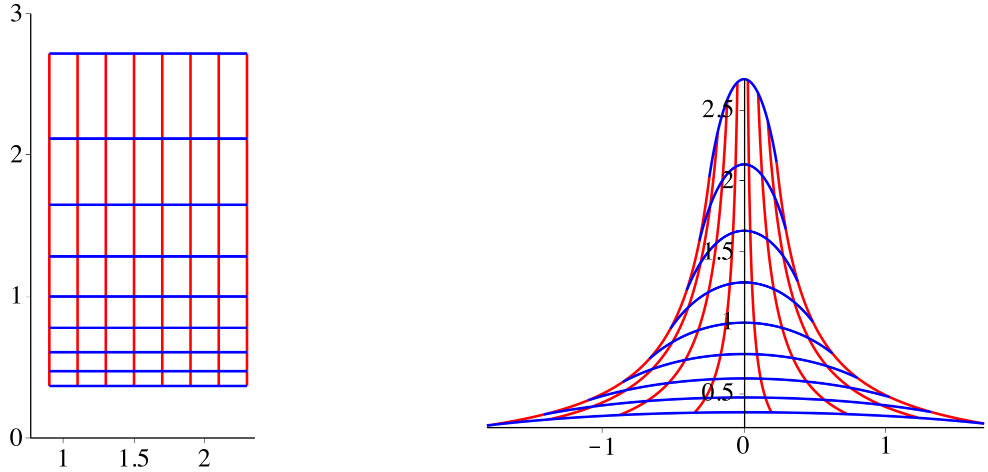


Figure 1: Some u-v coordinate lines in the set $(-\frac{\pi}{4}, \frac{\pi}{4}) \times (0, \infty)$, and their image under the map. Blue curves are portions of ellipses, and red curves are portions of hyperbolas.

since ϕ_α is a bijection, and this is an intersection of open subsets of $\mathbb{R}^n$, hence open. The same thing works for arbitrary unions, again since ϕ_α is a bijection. So we have a topology.

To show that each U_α is open, we need to show that for any β, the set $\phi_\beta[U_\alpha \cap U_\beta]$ is open, but this is precisely one of our assumptions. To show that each ϕ_α is continuous, we need to show that for any open set $V \subset \mathbb{R}^n$, the set $\phi_\alpha^{-1}[V]$ is open in M. But that set is open in M if and only if $\phi_\beta[\phi_\alpha^{-1}[V] \cap U_\beta]$ is open in $\mathbb{R}^n$. This latter set can be rewritten, again since ϕ_α is a bijection, as

$$\begin{aligned}
\phi_\beta[\phi_\alpha^{-1}[V] \cap U_\beta] &= \phi_\beta[\phi_\alpha^{-1}[V] \cap U_\alpha \cap U_\beta] \\
&= \phi_\beta\Big[\phi_\alpha^{-1}\big[V \cap \phi_\alpha[U_\alpha \cap U_\beta]\big]\Big] \\
&= (\phi_\alpha \circ \phi_\beta^{-1})\big[V \cap \phi_\alpha[U_\alpha \cap U_\beta]\big].
\end{aligned}$$

Since $\phi_\alpha \circ \phi_\beta^{-1}$ is assumed continuous, and since $\phi_\alpha[U_\alpha \cap U_\beta]$ is open by assumption, we know this set is open.