Math 6230 Homework #3
due Friday, February 8

Read "Introduction to Differential Geometry" through Chapter 7.1.

1. Consider the function $F: \mathbb{R}^3 \to \mathbb{R}$ given by

$$F(x, y, z) = x^2 + y^2 + z^2 + 2xyz,$$

and the level surfaces S_c consisting of the points satisfying $F(x, y, z) = c$ for $c > 0$.

   (a) Show using the Implicit Function Theorem that if $c \neq 1$, then we can always locally represent S_c as the graph of a smooth function of one of the variables in terms of the others.

   (b) What happens when $c = 1$? (Try plotting it.)

2. For the differential equation

$$\frac{dx}{dt} = x^2, \qquad x(0) = 1,$$

   use Picard iteration to obtain the approximate solutions up to $k = 2$. Check that $x(t) = \frac{1}{1-t}$ is the exact solution and compare your functions $\eta_k(t)$ to its Taylor series.

3. Consider the coordinate chart $(x, y) = F(u, v) = (v \cos u, \sin u / v)$. Find the largest open set U around $(u, v) = (0, 1)$ such that F is a diffeomorphism on U (i.e., F is smooth, invertible, and F^{-1} is also smooth). What is the image of U in the plane? What do the coordinate curves look like?

4. Suppose M is a set, I is some index set, and we have a collection of sets $U_\alpha \subset M$ and bijective functions ϕ_α which map U_α onto $\mathbb{R}^n$ for each $\alpha \in I$. Suppose that the union of all U_α is M. Define a set $\Omega \subset M$ to be open if and only if $\phi_\alpha[\Omega \cap U_\alpha]$ is open for every $\alpha \in I$. Check that this definition satisfies the conditions for a topology on M.

   If we further demand that whenever $U_\alpha \cap U_\beta$ is nonempty, the set $\phi_\alpha[U_\alpha \cap U_\beta]$ is open in $\mathbb{R}^n$ and the function $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta[U_\alpha \cap U_\beta] \subset \mathbb{R}^n \to \phi_\alpha[U_\alpha \cap U_\beta] \subset \mathbb{R}^n$ is a homeomorphism, show that each U_α is open in this topology, and that each ϕ_α is continuous in this topology.