Math 70900 Homework #2 Solutions

1. (a) Let V be a two-dimensional vector space, and consider the $(2,0)$ tensor $g\colon V\times V \to \mathbb{R}$ given in a basis $\{e_1, e_2\}$ by $g_{ij} = g(e_i, e_j) = \delta_{ij}$. If a new basis $\{f_1, f_2\}$ is given by $f_1 = 3e_1 - 4e_2$, $f_2 = -2e_1 + 3e_2$, find the coefficients $\tilde{g}_{ij}$ in the new basis.

   Solution: By definition, the coefficients are just what we get when we apply the tensor to the new basis vectors. That is,

$$\begin{aligned}\tilde{g}_{11} &= g(f_1, f_1) = g(3e_1 - 4e_2, 3e_1 - 4e_2) = 9 - 0 + 16 = 25\\ \tilde{g}_{12} &= g(f_1, f_2) = g(3e_1 - 4e_2, -2e_1 + 3e_2) = -6 + 0 + 0 - 12 = -18\\ \tilde{g}_{22} &= g(f_2, f_2) = g(-2e_1 + 3e_2, -2e_1 + 3e_2) = 4 - 0 + 9 = 13.\end{aligned}$$

   (b) Generally, if g is a $(2,0)$ tensor on an n-dimensional vector space and $g_{ij} = \delta_{ij}$ in a basis $\{e_1, \ldots, e_n\}$, what are the components $\tilde{g}_{ij}$ in a new basis $\{f_1, \ldots, f_n\}$, related to the e-basis by $f_i = \sum_j p_i^j e_j$ and $e_i = \sum_j q_i^j f_j$?

   Solution: The components are

$$\begin{aligned}\tilde{g}_{ij} = g(f_i, f_j) &= g\left(\sum_{k=1}^n p_i^k e_k, \sum_{\ell=1}^n p_j^\ell e_\ell\right)\\ &= \sum_{k=1}^n \sum_{\ell=1}^n p_i^k p_j^\ell g(e_k, e_\ell) = \sum_{k=1}^n \sum_{\ell=1}^n p_i^k p_j^\ell g_{k\ell}.\end{aligned}$$

2. A *symplectic form* on a vector space V is a 2-form ω that is nondegenerate, i.e., $\omega(u, v) = 0$ for all $v \in V$ implies that $u = 0$.

   (a) Which 2-forms are symplectic on a 2-dimensional vector space?

   Solution: Every 2-form on a 2-dimensional vector space can be written as $\omega = c\,\alpha^1 \otimes \alpha^2$ in terms of a dual basis. Write a vector u as $u = ae_1 + be_2$, and let's see what $\omega(u, e_1) = \omega(u, e_2) = 0$ imply. We have

$$\begin{aligned}\omega(u, e_1) &= \omega(ae_1 + be_2, e_1) = -b\omega(e_1, e_2) = -bc\\ \omega(u, e_2) &= \omega(ae_1 + be_2, e_2) = a\omega(e_1, e_2) = ac.\end{aligned}$$

   Now if $-bc = 0$ and $ac = 0$, then either $c = 0$ or $a = b = 0$. Hence as long as $c \neq 0$, the assumptions $\omega(u, e_1) = \omega(u, e_2) = 0$ imply $a = b = 0$ and thus $u = 0$. So any nonzero 2-form is symplectic.

   (b) Suppose V is 4-dimensional with basis $\{e_1, \ldots, e_4\}$ and dual basis $\{\alpha^1, \ldots, \alpha^4\}$. Write a general 2-form on V as

$$\omega = a\,\alpha^1 \wedge \alpha^2 + b\,\alpha^1 \wedge \alpha^3 + c\,\alpha^1 \wedge \alpha^4 + d\,\alpha^2 \wedge \alpha^3 + e\,\alpha^2 \wedge \alpha^4 + f\,\alpha^3 \wedge \alpha^4.$$

   What is the condition on the coefficients to make this a symplectic form?

Solution: Let $u = pe_1 + qe_2 + re_3 + se_4$. If $\omega(u, e_k) = 0$ for every k, then we get the four equations

$$\begin{aligned}
\omega(u, e_1) &= q\omega(e_2, e_1) + r\omega(e_3, e_1) + s\omega(e_4, e_1) = -aq - br - cs = 0 \\
\omega(u, e_2) &= p\omega(e_1, e_2) + r\omega(e_3, e_2) + s\omega(e_4, e_2) = ap - dr - es = 0 \\
\omega(u, e_3) &= p\omega(e_1, e_3) + q\omega(e_2, e_3) + s\omega(e_4, e_3) = bp + dq - fs = 0 \\
\omega(u, e_4) &= p\omega(e_1, e_4) + q\omega(e_2, e_4) + r\omega(e_3, e_4) = cp + eq + fr = 0.
\end{aligned}$$

This translates into the matrix system $Au = 0$ where

$$Au = \begin{pmatrix} 0 & -a & -b & -c \\ a & 0 & -d & -e \\ b & d & 0 & -f \\ c & e & f & 0 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and the question is when this forces the vector u to be 0. Obviously that's true if and only if the determinant of this 4×4 matrix is nonzero, and Maple helpfully tells me this determinant is

$$\det A = a^2 f^2 + 2adfc - 2aebf + b^2e^2 - 2bedc + d^2c^2 = (af + dc - be)^2.$$

So the condition is $af + dc - be \neq 0$.

(c) Prove that ω is a symplectic form on a 4-dimensional vector space if and only if $\omega \wedge \omega$ is nonzero.

Solution: We already know the condition for nondegeneracy, so we just need to compute $\omega \wedge \omega$.

Notice that since ω is a 2-form, we know $\omega \wedge \omega$ is a 4-form, and since the vector space is 4-dimensional, the only possibility is a multiple of $\alpha^1 \wedge \alpha^2 \wedge \alpha^3 \wedge \alpha^4$. This means that when we compute $\omega \wedge \omega$ by distributing terms, each term in the first ω will give zero when applied to five out of six terms in the second ω. For example the term $a\,\alpha^1 \wedge \alpha^2$ gives zero when wedged with anything in ω except for $f\,\alpha^3 \wedge \alpha^4$, when it gives $af\,\alpha^1 \wedge \alpha^2 \wedge \alpha^3 \wedge \alpha^4$.

Expanding the wedge product and transposing until every term is a multiple of $\alpha^1 \wedge \alpha^2 \wedge \alpha^3 \wedge \alpha^4$, we obtain

$$\omega \wedge \omega = 2(af + dc - be)\,\alpha^1 \wedge \alpha^2 \wedge \alpha^3 \wedge \alpha^4.$$

Hence $\omega \wedge \omega \neq 0$ if and only if $af + dc - be \neq 0$, which is precisely the condition for ω to be nondegenerate.

3. (a) If ω is any 2-form on a 3-dimensional vector space, prove that there are 1-forms α and β such that $\omega = \alpha \wedge \beta$. (Hint: if you work this out in a basis, it's essentially the same statement as "any vector in $\mathbb{R}^3$ is the cross product of two other vectors.")

Solution: As hinted, expand in a basis. Express our unknown 1-forms as $\alpha = p_1\alpha^1 + p_2\alpha^2 + p_3\alpha^3$ and $\beta = q_1\alpha^1 + q_2\alpha^2 + q_3\alpha^3$; then we have

$$\alpha \wedge \beta = (p_1q_2 - p_2q_1)\,\alpha^1 \wedge \alpha^2 + (p_2q_3 - p_3q_2)\,\alpha^2 \wedge \alpha^3 + (p_3q_1 - p_1q_3)\,\alpha^3 \wedge \alpha^1.$$

Now our given ω can be expressed as

$$\omega = r_1\, \alpha^2 \wedge \alpha^3 + r_2\, \alpha^3 \wedge \alpha^1 + r_3\, \alpha^1 \wedge \alpha^2,$$

so matching these up gives the equations

$$p_1q_2 - p_2q_1 = r_3, \qquad p_2q_3 - p_3q_2 = r_1, \qquad p_3q_1 - p_1q_3 = r_2,$$

which is exactly the statement that, as vectors in $\mathbb{R}^3$ with the usual cross-product, we have

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \times \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}.$$

This makes it easier to visualize what's going on.

To actually prove the statement, let $\mathbf{r}$ be any vector in $\mathbb{R}^3$ and choose any unit vector $\mathbf{p}$ which is orthogonal to $\mathbf{r}$. Define $\mathbf{q} = \mathbf{r} \times \mathbf{p}$; then by standard vector calculus we have

$$\mathbf{p} \times \mathbf{q} = \mathbf{p} \times (\mathbf{r} \times \mathbf{p}) = (\mathbf{p} \cdot \mathbf{p})\mathbf{r} - (\mathbf{p} \cdot \mathbf{r})\mathbf{p} = \mathbf{r}.$$

(b) Use this to show that no 2-form on a 3-dimensional vector space can ever be nondegenerate (as defined in the previous problem).

Solution: Let ω be a 2-form with $\omega = \alpha \wedge \beta$, and assume α and β are both nonzero 1-forms. Let $V_1 \subset V$ be the kernel of α, and let $V_2 \subset V$ be the kernel of β. Then V_1 is a two-dimensional subspace and V_2 is a two-dimensional subspace, and their intersection is at least a one-dimensional subspace. So there is a nonzero vector u such that $\alpha(u) = \beta(u) = 0$. Thus for every vector $v \in V$ we have

$$\omega(u, v) = (\alpha \wedge \beta)(u, v) = \alpha(u)\beta(v) - \alpha(v)\beta(u) = 0.$$

Hence ω is degenerate.

(c) Give an explicit example of a 2-form on a 4-dimensional vector space which cannot be written as a product $\omega = \alpha \wedge \beta$.

Solution: By the same reasoning as in part (b), any 2-form that can be written as $\omega = \alpha \wedge \beta$ on a space of dimension at least three must be degenerate. Therefore any nondegenerate 2-form cannot be written as a product.

In the previous problem we found a condition for nondegeneracy in terms of the coefficients $\{a, b, c, d, e, f\}$: that $af + dc - be \neq 0$. The simplest choice is $a = f = 1$ and $b = c = d = e = 0$, so that

$$\omega = \alpha^1 \wedge \alpha^2 + \alpha^3 \wedge \alpha^4.$$

4. Suppose V is two-dimensional and W is three-dimensional, with a linear transformation $T\colon V \to W$ expressed in some basis $\{e_1, e_2\}$ and $\{f_1, f_2, f_3\}$ as $T(e_1) = 4f_1 - 5f_2 + f_3$ and $T(e_2) = 2f_1 + 7f_3$. Let ω be a 2-form satisfying $\omega(f_1, f_2) = -2$, $\omega(f_2, f_3) = 5$, and $\omega(f_3, f_1) = 4$. Compute $T^*\omega$.

Solution: $T^*\omega$ is a 2-form on a two-dimensional space, which means it is completely determined by $(T^*\omega)(e_1, e_2)$. By definition we have

$$\begin{aligned}(T^*\omega)(e_1,e_2) &= \omega\big(T(e_1),T(e_2)\big)\\ &= \omega(4f_1 - 5f_2 + f_3, 2f_1 + 7f_3)\\ &= 28\omega(f_1,f_3) - 10\omega(f_2,f_1) - 35\omega(f_2,f_3) + 2\omega(f_3,f_1)\\ &= -28\cdot 4 - 10\cdot 2 - 35\cdot 5 + 2\cdot 4\\ &= -299.\end{aligned}$$

Therefore we must have

$$T^*\omega = -299\,\alpha^1 \wedge \alpha^2.$$

5. Compute explicitly the map $F(A) = A^\dagger A$, from the space of all 2×2 matrices (equivalent to $\mathbb{R}^4$) to the space of symmetric 2×2 matrices (equivalent to $\mathbb{R}^3$). Then find its derivative $DF(A)$.

   At what matrices A does $DF(A)$ have maximal rank?

   Solution: Let $A = \left(\begin{smallmatrix} w & x\\ y & z\end{smallmatrix}\right)$ and $A^\dagger A = \left(\begin{smallmatrix} p & q\\ q & r\end{smallmatrix}\right)$. Then the map is

$$(p,q,r) = F(w,x,y,z) = (w^2+x^2, wy+xz, y^2+z^2).$$

   Its derivative is

$$DF(w,x,y,z) = \begin{pmatrix} p_w & p_x & p_y & p_z\\ q_w & q_x & q_y & q_z\\ r_w & r_x & r_y & r_z\end{pmatrix} = \begin{pmatrix} 2w & 2x & 0 & 0\\ y & z & w & x\\ 0 & 0 & 2y & 2z\end{pmatrix}.$$

   We want to know when this matrix has rank 3, and we will use Proposition 3.3.6 to do it. First compute the left 3×3 determinant to get $4y(wz-xy)$; then compute the right 3×3 determinant to get $4x(wz-xy)$.

   If $wz - xy \neq 0$, then the only way both of these determinants are zero is if $x = y = 0$. Now if $x = y = 0$ and $wz - xy \neq 0$, then $wz \neq 0$. The matrix DF becomes

$$DF(w,0,0,z) = \begin{pmatrix} 2w & 0 & 0 & 0\\ 0 & z & w & 0\\ 0 & 0 & 0 & 2z\end{pmatrix}.$$

   Consider the determinant obtained by deleting the third column: it's $4w^2z \neq 0$, and so in this case the rank is maximal. We have shown that if $wz - xy \neq 0$, then $DF(w,x,y,z)$ has maximal rank.

   Suppose $wz - xy = 0$. If $w \neq 0$, row reduction gives

$$DF(w,x,y,z) \sim \begin{pmatrix} 2w & 2x & 0 & 0\\ 0 & 0 & w & x\\ 0 & 0 & 2y & 2z\end{pmatrix} \sim \begin{pmatrix} 2w & 2x & 0 & 0\\ 0 & 0 & w & x\\ 0 & 0 & 0 & 0\end{pmatrix},$$

   which has rank two.

On the other hand if $w = 0$ and $wz - xy = 0$, then we must have $xy = 0$, so either $x = 0$ or $y = 0$. If $x = 0$ then

$$DF(0, 0, y, z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ y & z & 0 & 0 \\ 0 & 0 & y & z \end{pmatrix},$$

which has rank two if either $y \neq 0$ or $z \neq 0$. If $y = 0$ then

$$DF(0, x, 0, z) = \begin{pmatrix} 0 & 2x & 0 & 0 \\ 0 & z & 0 & x \\ 0 & 0 & 0 & 2z \end{pmatrix},$$

which has rank two if either $x \neq 0$ or $z \neq 0$.

In summary, if $wz - xy = 0$ but at least one of $\{w, x, y, z\}$ is not zero, then $DF(w, x, y, z)$ has rank two. However if all entries are zero, then $DF(0, 0, 0, 0)$ is the zero matrix which has rank zero.

6. In the proof of Theorem 5.2.2 (the Implicit Function Theorem), it was claimed that if F has infinitely many continuous derivatives, then so does G. Compute $G'(x)$ and $G''(x)$ in the case $k = n = 1$.

   Solution: In the case $k = n = 1$, we have

   $$F(x, G(x)) = 0$$

   for a function $G \colon U \subset \mathbb{R} \to \mathbb{R}$. Differentiating using the chain rule, we get

   $$F_x(x, G(x)) + F_y(x, G(x))G'(x) = 0,$$

   and solving gives

   $$G'(x) = -\frac{F_x(x, G(x))}{F_y(x, G(x))},$$

   which exists in a neighborhood of the point (x_0, y_0) since $F_y(x_0, y_0) \neq 0$ by assumption.

   Differentiating again, we get

   $$F_{xx}(x, G(x)) + 2F_{xy}(x, G(x))G'(x) + F_{yy}(x, G(x))G'(x)^2 + F_y(x, G(x))G''(x) = 0.$$

   Solving for $G''(x)$ and using our answer for $G'(x)$, we get

   $$G''(x) = -\frac{F_y^2 F_{xx} - 2F_x F_y F_{xy} + F_x^2 F_{yy}}{F_y^3},$$

   where all functions are evaluated at $(x, G(x))$.

7. Suppose that the solution of $x'(t) = t^3 + x(t)^3$ with initial condition $x(0) = a$ is denoted by $\Gamma(t, a)$. Find a formula for $Z(t, a) := \frac{\partial \Gamma}{\partial a}(t, a)$ in terms of the function Γ.

Solution: We have

$$\frac{\partial \Gamma}{\partial t}(t,a) = t^3 + \Gamma(t,a)^3$$

for all t and a, and thus we can differentiate with respect to a to get an equation for Z: we have

$$\frac{\partial Z}{\partial t}(t,a) = 3\Gamma(t,a)^2 Z(t,a).$$

Since $\Gamma(0,a) = a$ for all a, differentiating this equation with respect to a gives $Z(0,a) = 1$.

The solution of this linear ODE with this initial condition is

$$Z(t,a) = \exp\left(3\int_0^t \Gamma(s,a)^2\, ds\right).$$