

Read “Introduction to Differential Geometry” through Chapter 5.

- Let V be a two-dimensional vector space, and consider the $(2, 0)$ tensor $g: V \times V \rightarrow \mathbb{R}$ given in a basis $\{e_1, e_2\}$ by $g_{ij} = g(e_i, e_j) = \delta_{ij}$. If a new basis $\{f_1, f_2\}$ is given by $f_1 = 3e_1 - 4e_2$, $f_2 = -2e_1 + 3e_2$, find the coefficients \tilde{g}_{ij} in the new basis.
 - Generally, if g is a $(2, 0)$ tensor on an n -dimensional vector space and $g_{ij} = \delta_{ij}$ in a basis $\{e_1, \dots, e_n\}$, what are the components \tilde{g}_{ij} in a new basis $\{f_1, \dots, f_n\}$, related to the e -basis by $f_i = \sum_j p_i^j e_j$ and $e_i = \sum_j q_i^j f_j$?
- A *symplectic form* on a vector space V is a 2-form ω that is nondegenerate, i.e., $\omega(u, v) = 0$ for all $v \in V$ implies that $u = 0$.
 - Which 2-forms are symplectic on a 2-dimensional vector space?
 - Suppose V is 4-dimensional with basis $\{e_1, \dots, e_4\}$ and dual basis $\{\alpha^1, \dots, \alpha^4\}$. Write a general 2-form on V as
$$\omega = a \alpha^1 \wedge \alpha^2 + b \alpha^1 \wedge \alpha^3 + c \alpha^1 \wedge \alpha^4 + d \alpha^2 \wedge \alpha^3 + e \alpha^2 \wedge \alpha^4 + f \alpha^3 \wedge \alpha^4.$$
What is the condition on the coefficients to make this a symplectic form?
 - Prove that ω is a symplectic form on a 4-dimensional vector space if and only if $\omega \wedge \omega$ is nonzero.
- If ω is any 2-form on a 3-dimensional vector space, prove that there are 1-forms α and β such that $\omega = \alpha \wedge \beta$. (Hint: if you work this out in a basis, it's essentially the same statement as “any vector in \mathbb{R}^3 is the cross product of two other vectors.”)
 - Use this to show that no 2-form on a 3-dimensional vector space can ever be nondegenerate (as defined in the previous problem).
 - Give an explicit example of a 2-form on a 4-dimensional vector space which cannot be written as a product $\omega = \alpha \wedge \beta$.
- Suppose V is two-dimensional and W is three-dimensional, with a linear transformation $T: V \rightarrow W$ expressed in some basis $\{e_1, e_2\}$ and $\{f_1, f_2, f_3\}$ as $T(e_1) = 4f_1 - 5f_2 + f_3$ and $T(e_2) = 2f_1 + 7f_3$. Let ω be a 2-form satisfying $\omega(f_1, f_2) = -2$, $\omega(f_2, f_3) = 5$, and $\omega(f_3, f_1) = 4$. Compute $T^*\omega$.
- Compute explicitly the map $F(A) = A^\dagger A$, from the space of all 2×2 matrices (equivalent to \mathbb{R}^4) to the space of symmetric 2×2 matrices (equivalent to \mathbb{R}^3). Then find its derivative $DF(A)$.

At what matrices A does $DF(A)$ have maximal rank?
- In the proof of Theorem 5.2.2 (the Implicit Function Theorem), it was claimed that if F has infinitely many continuous derivatives, then so does G . Compute $G'(x)$ and $G''(x)$ in the case $k = n = 1$.
- Suppose that the solution of $x'(t) = t^3 + x(t)^3$ with initial condition $x(0) = a$ is denoted by $\Gamma(t, a)$. Find a formula for $Z(t, a) := \frac{\partial \Gamma}{\partial a}(t, a)$ in terms of the function Γ .