Math 70900 Homework #1 Solutions

Read "Introduction to Differential Geometry" up to Chapter 4.

1. Suppose you lived on a small sphere of radius R, so that spherical geometry was more natural than planar geometry. What formulas would children be taught in school instead of the Euclidean formulas $C = 2\pi r$ (for circumference of a circle) and $A = \pi r^2$ (for area of a circle)?

   Solution: The circumference is easy. Look at the sphere from the side as in Figure 1.

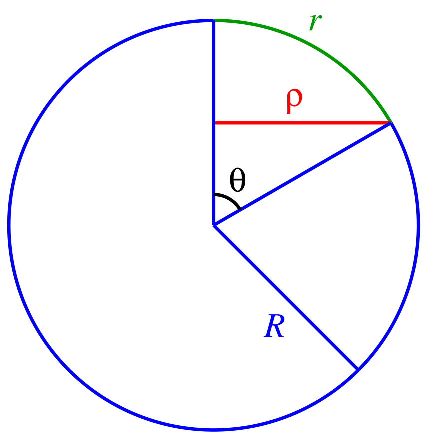

Figure 1: A circle in the sphere: the residents measure r, but the actual circumference is determined by the Euclidean result for the circle of radius ρ.

It has some radius R. An arc length r can be measured from the north pole (traced out by angle θ), and a circle can be drawn on the sphere by the residents. In Euclidean 3-space that circle will have radius ρ, although the people on the sphere cannot measure that directly.

From trigonometry we have the relations $r = R\theta$ and $R\sin\theta = \rho$. The circumference of the circle is $C = 2\pi\rho$, and in terms of the observable quantity r and the parameter R, the formula is

$$C = 2\pi R \sin\left(\frac{r}{R}\right).$$

Now to compute the area of the circle, there are two ways to do it. The more direct vector calculus method would be to view it as a surface area problem in three dimensions. We write $z = f(x,y) = \sqrt{R^2 - x^2 - y^2}$, and use the surface area formula

$$A = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy = \iint_D \frac{R\, dx\, dy}{\sqrt{R^2 - x^2 - y^2}} = \int_0^{2\pi} \int_0^{\rho} \frac{Rp\, dp}{\sqrt{R^2 - p^2}}\, dp$$
$$= -2\pi R\sqrt{R^2 - p^2}\Big|_0^{\rho} = 2\pi R^2 \left[1 - \cos\left(\frac{r}{R}\right)\right].$$

The more elegant way is to just notice that area is the integral of circumference with respect to the radius, which gives the same answer.

Notice that it's easy to check these formulas in the special cases where we know the answer ($r = \frac{R\pi}{2}$ and $r = R\pi$). It's also easy to see via Taylor expansions for small r that we get the Euclidean formula back to lowest order.

2. Recall that in the hyperbolic plane (the upper half-plane model $H = \{(x, y) : y > 0\}$) of hyperbolic geometry, "lines" are either semicircles centered on the x-axis or vertical lines, as shown in Figure 1.5.

   Show that for any two points (x_1, y_1) and (x_2, y_2) on the hyperbolic plane there is a unique hyperbolic line passing through the points; compute it explicitly.

   Solution: We are looking for either a circle centered on the x-axis or a vertical line. First we look for a circle: we want

$$(x_1 - a)^2 + y_1^2 = r^2 \quad \text{and} \quad (x_2 - a)^2 + y_2^2 = r^2$$

   for some $a \in \mathbb{R}$ and $r > 0$. Eliminating r^2 first, if $x_1 \neq x_2$ we get

$$a = \frac{y_2^2 - y_1^2 + x_2^2 - x_1^2}{2(x_2 - x_1)} \quad \text{and} \quad r^2 = y_1^2 + \left(\frac{(x_1 - x_2)^2 + y_2^2 - y_1^2}{2(x_2 - x_1)} \right)^2.$$

   On the other hand if $x_1 = x_2$ then we get a unique vertical line $x = x_1$ passing through.

3. In standard spherical coordinates on S^2, with $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$, and $z = \cos\theta$, show that $\sin^2\theta$ is a smooth function on the sphere (because it is the restriction of a smooth function on $\mathbb{R}^3$), but that $\sin^2\phi$ is not a smooth function on the sphere. (Hint: what does it look like near the north pole in (x, y) coordinates?)

   Solution: Since $x^2 + y^2 = \sin^2\theta$ for any point (x, y, z) on the sphere, the function $\sin^2\theta$ is the restriction of the function $f \colon \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x^2 + y^2$, which is obviously C^∞. So the restriction is also smooth.

   On the other hand, we have

$$\sin^2\phi = \frac{y^2}{\sin^2\theta} = \frac{y^2}{x^2 + y^2}.$$

   Using (x, y) as coordinates, the north pole corresponds to $x = y = 0$, and this function is not continuous at the origin. Thus in particular it is not smooth.

4. Consider two possible bases for $\mathbb{R}^2$:

$$e_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

   vs.

$$f_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad \text{and} \quad f_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

(a) Find the transformation matrix P such that $f_i = \sum_{j=1}^{2} p_i^j e_j$, and the transformation matrix Q such that $e_i = \sum_{j=1}^{2} q_i^j f_j$.
Solution:
Comparing components of f_1 and f_2, we need to solve the equations

$$\begin{aligned} 2 &= 5p_1^1 + 2p_1^2 & \qquad -3 &= 2p_1^1 + p_1^2 \\ 1 &= 5p_2^1 + 2p_2^2 & \qquad -2 &= 2p_1^2 + p_2^2. \end{aligned}$$

It is easy to see that $p_1^1 = 8$, $p_1^2 = -19$, $p_2^1 = 5$, and $p_2^2 = -12$.
Then Q is the inverse matrix, so that $q_1^1 = 12$, $q_1^2 = -19$, $q_2^1 = 5$, $q_2^2 = -8$.

(b) How would you express the vector $v = 7e_1 - 3e_2$ in the $\{f\}$-basis?
Solution:
Easy enough. Just write

$$v = 7(q_1^1 f_1 + q_1^2 f_2) - 3(q_2^1 f_1 + q_2^2 f_2) = 69 f_1 - 109 f_2.$$

(c) Compute explicitly the covectors α^1 and α^2 (satisfying $\alpha^i(e_j) = \delta_j^i$), and the covectors β^1 and β^2 satisfying $\beta^i(f_j) = \delta_j^i$.
Solution:
We write $\alpha^1 = (a \quad b)$ and $\alpha^2 = (c \quad d)$; then these numbers must satisfy

$$\begin{aligned} 5a + 2b &= 1 & \qquad 2a + b &= 0 \\ 5c + 2d &= 0 & \qquad 2c + d &= 1. \end{aligned}$$

The solution is $\alpha^1 = (1 \quad -2)$ and $\alpha^2 = (-2 \quad 5)$.
Similarly the βs are given by $\beta^1 = (2 \quad 1)$ and $\beta^2 = (-3 \quad -2)$.

5. A linear operator $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is given in the standard basis $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ by

$$T = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -2 \\ -1 & 0 & -1 \end{pmatrix}.$$

(a) If you think of the domain and range of T as different vector spaces, show how to change the bases of both to get T in reduced row echelon form. (Describe the new bases explicitly.)
Solution:
The matrix equation says that $T(e_1) = f_1 - f_3$, $T(e_2) = -f_1 + 2f_2$, and $T(e_3) = 2f_1 - 2f_2 - f_3$.
Doing the row reduction operations we see that T is similar to $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. (This is as simple as we can get it without doing column operations, i.e., without changing the basis of the domain.)

This corresponds to the new basis of the range $\tilde{f}_1 = f_1 - f_3$, $\tilde{f}_2 = -f_1 + 2f_2$, while $\tilde{f}_3$ can be any vector that is linearly independent from the other two (for example, $\tilde{f}_3 = f_1$). Then we have $T(e_1) = \tilde{f}_1$, $T(e_2) = \tilde{f}_2$, and $T(e_3) = \tilde{f}_1 - \tilde{f}_2$.

If we also change the domain basis, e.g., to $\tilde{e}_1 = e_1$, $\tilde{e}_2 = e_2$, and $\tilde{e}_3 = e_3 - e_1 + e_2$, then things get even simpler: we have $T(\tilde{e}_1) = \tilde{f}_1$, $T(\tilde{e}_2) = \tilde{f}_2$, and $T(\tilde{e}_3) = 0$, corresponding to the reduced matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(b) If you think of the domain and range of T as the same vector space, show how to change the basis to get T in Jordan form.

Solution: To find the Jordan form, we need to compute the characteristic polynomial

$$\det(\lambda I - T) = \lambda^3 - 2\lambda^2 + \lambda.$$

The roots of this are $\lambda = 0$, $\lambda = 1$, and $\lambda = 1$ (a double root).

An eigenvector corresponding to $\lambda = 0$ is $v_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$, while an eigenvector corresponding to $\lambda = 1$ is $v_2 = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$. Unfortunately we don't have a third eigenvector corresponding to the degenerate eigenvalue $\lambda = 1$, so instead we use the Jordan block form, corresponding to asking for a vector v_3 satisfying $Tv_3 = v_3 + v_2$. This can be done; one such vector is $v_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$.

In the new basis $\{v_1, v_2, v_3\}$, T takes the form $T(v_1) = 0$, $T(v_2) = v_2$, and $T(v_3) = v_3 + v_2$, corresponding to the Jordan form matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

6. Prove directly, imitating Proposition 3.3.1, that if $T\colon V \to V$ is a linear operator, then the number $\mathrm{Tr}(T^2) = \sum_{i=1}^n \sum_{j=1}^n T_i^j T_j^i$ does not depend on choice of basis.

Solution: Suppose $T(e_i) = \sum_{j=1}^n T_i^j e_j$ in the given basis $\{e_1, \ldots, e_n\}$, and that in some new basis $\{f_1, \ldots, f_n\}$ we have $T(f_i) = \sum_{j=1}^n \tilde{T}_i^j f_j$. If the bases are related by

$$f_i = \sum_{j=1}^n p_i^j e_j, \qquad e_i = \sum_{j=1}^n q_i^j f_j,$$

then we have

$$T(f_i) = \sum_{\ell=1}^n p_i^\ell T(e_\ell) = \sum_{\ell=1}^n \sum_{k=1}^n p_i^\ell T_\ell^k e_k = \sum_{\ell=1}^n \sum_{k=1}^n \sum_{j=1}^n p_i^\ell T_\ell^k q_k^j f_j,$$

and we conclude that

$$\tilde{T}_i^j = \sum_{\ell=1}^n \sum_{k=1}^n p_i^\ell T_\ell^k q_k^j.$$

Applying T twice to a basis vector, we get

$$T^2(e_i) = T\left(\sum_{k=1}^n T_i^k e_k\right) = \sum_{k=1}^n T_i^k T(e_k) = \sum_{k=1}^n \sum_{j=1}^n T_i^k T_k^j e_j,$$

and thus if we write $U = T^2$ and its coefficients as U_i^j, we get

$$U_i^j = \sum_{k=1}^{n} T_i^k T_k^j$$

and

$$\tilde{U}_i^j = \sum_{k=1}^{n} \tilde{T}_i^k \tilde{T}_k^j = \sum_{k=1}^{n} \sum_{\ell=1}^{n} \sum_{m=1}^{n} \sum_{a=1}^{n} \sum_{b=1}^{n} p_i^\ell T_\ell^m q_m^k p_k^a T_a^b q_b^j.$$

Then the trace in the e basis is

$$\mathrm{Tr}(T^2) = \sum_{i=1}^{n} U_i^i = \sum_{k=1}^{n} \sum_{i=1}^{n} T_i^k T_k^i,$$

while the trace in the f basis is

$$\begin{aligned}
\mathrm{Tr}(T^2) = \sum_{i=1}^{n} \tilde{U}_i^i &= \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \sum_{m=1}^{n} \sum_{a=1}^{n} \sum_{b=1}^{n} p_i^\ell T_\ell^m q_m^k p_k^a T_a^b q_b^i \\
&= \sum_{\ell=1}^{n} \sum_{m=1}^{n} \sum_{a=1}^{n} \sum_{b=1}^{n} T_\ell^m T_a^b \left(\sum_{k=1}^{n} q_m^k p_k^a \right) \left(\sum_{i=1}^{n} p_i^\ell q_b^i \right) \\
&= \sum_{\ell=1}^{n} \sum_{m=1}^{n} \sum_{a=1}^{n} \sum_{b=1}^{n} T_\ell^m T_a^b \delta_m^a \delta_b^\ell \\
&= \sum_{\ell=1}^{n} \sum_{m=1}^{n} T_\ell^m T_m^\ell,
\end{aligned}$$

and this is the same as the formula in the e basis (with different dummy indices).