Math 70900 Homework #10 Solutions

1. Let
$$\omega = \frac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$$
on $\mathbb{R}^3\backslash\{(0,0,0)\}$. Show that ω is closed but not exact. Hint: to show it's not exact, integrate it over a parametrized 2-sphere and obtain a nonzero number.

Solution: To show ω is closed, we just perform a computation: if $\omega = u(x,y,z)\,dy\wedge dz + v(x,y,z)\,dz\wedge dx + w(x,y,z)\,dx\wedge dy$, then

$$\begin{aligned} d\omega &= \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dx\wedge dy\wedge dz \\ &= \left(\frac{\partial}{\partial x}\Big(\frac{x}{(x^2+y^2+z^2)^{3/2}}\Big) + \frac{\partial}{\partial y}\Big(\frac{y}{(x^2+y^2+z^2)^{3/2}}\Big) + \frac{\partial}{\partial z}\Big(\frac{z}{(x^2+y^2+z^2)^{3/2}}\Big)\right) dx\wedge dy\wedge dz \\ &= \left(\frac{3}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}\,\frac{2x^2+2y^2+2z^2}{(x^2+y^2+z^2)^{5/2}}\right) dx\wedge dy\wedge dz \\ &= 0. \end{aligned}$$

Now assume ω is exact. Then $\omega = d\alpha$ for some 1-form α on $\mathbb{R}^3\backslash\{(0,0,0)\}$. Furthermore if c is any 2-chain which maps into $\mathbb{R}^3$ without hitting the origin, then by Stokes' Theorem we have
$$\int_c \omega = \int_c d\alpha = \int_{\partial c} \alpha.$$
In particular if $\partial c = 0$ then $\int_c \omega = 0$. To obtain a contradiction, we will therefore find a 2-cube c with empty boundary such that $\int_c \omega \neq 0$.

Take $c = \big(\sin(\pi u)\cos(2\pi v), \sin(\pi u)\sin(2\pi v), \cos(\pi u)\big)$ defined on $[0,1]\times[0,1]$. Then
$$\begin{aligned} \partial c(t) &= c(1,t) - c(t,1) - c(0,t) + c(t,0) \\ &= (0,0,-1) - \big(\sin(\pi t), 0, \cos(\pi t)\big) - (0,0,1) + \big(\sin(\pi t), 0, \cos(\pi t)\big) \\ &= (0,0,-1) - (0,0,1). \end{aligned}$$

This is not exactly zero, but it's equivalent to zero: since it's degenerate (the push-forward c_* is always zero and hence the pull-back $c^\#$ is always zero) we will have $\int_{\partial c}\alpha = 0$ for any 1-form α.

Now actually compute $\int_c \omega$ for this ω and this c. We get
$$\begin{aligned} c^\# dx &= \pi\cos(\pi u)\cos(2\pi v)\,du - 2\pi\sin(\pi u)\sin(2\pi v)\,dv \\ c^\# dy &= \pi\cos(\pi u)\sin(2\pi v)\,du + 2\pi\sin(\pi u)\cos(2\pi v)\,dv \\ c^\# dz &= -\pi\sin(\pi u)\,du. \end{aligned}$$

Therefore we have
$$\begin{aligned} c^\#(dy\wedge dz) &= 2\pi^2\sin^2(\pi u)\cos(2\pi v)\,du\wedge dv \\ c^\#(dz\wedge dx) &= 2\pi^2\sin^2(\pi u)\sin(2\pi v)\,du\wedge dv \\ c^\#(dx\wedge dy) &= 2\pi^2\sin(\pi u)\cos(\pi u)\,du\wedge dv. \end{aligned}$$

Finally we obtain

$$\begin{aligned}
&c^{\#}(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy) \\
&\qquad = 2\pi^2 \Big(\sin^3(\pi u)\cos^2(2\pi v) + \sin^3(\pi u)\sin^2(2\pi v) + \sin(\pi u)\cos^2(\pi u) \Big)\, du \wedge dv \\
&\qquad = 2\pi^2 \sin(\pi u)\, du \wedge dv.
\end{aligned}$$

This is also equal to $c^{\#}\omega$ since $x^2 + y^2 + z^2 = 1$ on the image of c. Hence our integral is

$$\int_c \omega = \int_0^1 \int_0^1 2\pi^2 \sin(\pi u)\, du\, dv = 4\pi.$$

This isn't zero, so ω can't be exact.

2. (a) Suppose α is a closed k-form and β is an exact ℓ-form; show that $\alpha \wedge \beta$ is an exact $(k+\ell)$-form.

   Solution: By assumption we have $d\alpha = 0$ and $\beta = d\gamma$ for some $(\ell - 1)$-form γ. Therefore

$$\alpha \wedge \beta = \alpha \wedge d\gamma = (-1)^k d(\alpha \wedge \gamma) - (-1)^k d\alpha \wedge \gamma = d((-1)^k \alpha \wedge \gamma),$$

   and $\alpha \wedge \beta$ is exact.

   (b) Now consider the cohomology quotient spaces $H^k(M)$, where we say that two closed forms α_1 and α_2 are equivalent, $\alpha_1 \equiv \alpha_2$, if $\alpha_1 - \alpha_2 = d\phi$ for some $(k-1)$-form ϕ. Show that if $\alpha_1 \equiv \alpha_2$ as closed k-forms and $\beta_1 \equiv \beta_2$ as closed ℓ-forms, then $\alpha_1 \wedge \beta_1 \equiv \alpha_2 \wedge \beta_2$ as closed $(k+\ell)$-forms. The product induced on the cohomology spaces by the wedge is called the *cup product on de Rham cohomology.*

   Solution: We suppose $\alpha_1 = \alpha_2 + d\phi$ and $\beta_1 = \beta_2 + d\gamma$ for some $(k-1)$-form ϕ and $(\ell - 1)$-form γ. Now we compute

$$\begin{aligned}
\alpha_1 \wedge \beta_1 - \alpha_2 \wedge \beta_2 &= (\alpha_2 + d\phi) \wedge (\beta_2 + d\gamma) - \alpha_2 \wedge \beta_2 \\
&= d\phi \wedge \beta_2 + \alpha_2 \wedge d\gamma + d\phi \wedge d\gamma \\
&= d\Big(\phi \wedge \beta_2 + (-1)^k \alpha_2 \wedge \gamma + \phi \wedge d\gamma \Big)
\end{aligned}$$

   since α_2 and β_2 are closed. Thus $\alpha_1 \wedge \beta_1 \equiv \alpha_2 \wedge \beta_2$.

3. A smooth closed curve $\gamma\colon [0,1] \to M$ (with $\gamma(0) = \gamma(1)$) is called *smoothly contractible* if there is a point $p \in M$ and a smooth map $H\colon [0,1] \times [0,1] \to M$ such that

   - $H(0,t) = p$ for all $t \in [0,1]$;
   - $H(1,t) = \gamma(t)$ for all $t \in [0,1]$;
   - $H(s,0) = H(s,1)$ for all $s \in [0,1]$.

   If γ is smoothly contractible, show that $\gamma = \partial H$. Conversely if γ is the boundary of a disc (that is, a map $c\colon [0,1] \times [0,1] \to M$ of the form $c(r,\theta) = b\big(r\cos(2\pi\theta), r\sin(2\pi\theta)\big)$ for some smooth $b\colon \mathbb{R}^2 \to M$), show that γ is smoothly contractible.

Solution: Assuming such a homotopy, we have

$$\begin{aligned}\partial H(t) &= H(1,t) - H(t,1) - H(0,t) + H(t,0)\\ &= \gamma(t) - H(t,1) + H(t,0) - p.\end{aligned}$$

Since $H(t,1) = H(t,0)$ by assumption, we conclude that $\partial H(t) = \gamma(t) - p$. Now p is just a point, and the integral of any 1-form over a single point is zero, so γ is equivalent to ∂H from the point of view of integration.

Conversely suppose that $\partial c(t) = \gamma(t)$ for c of the form specified. We have

$$\partial c(t) = c(1,t) - c(t,1) - c(0,t) + c(t,0) = b(\cos{(2\pi t)}, \sin{(2\pi t)}) - b(t,0) + b(0,0) + b(t,0).$$

Thus the statement is that, up to an irrelevant point, $\gamma(t) = b\big(\cos{(2\pi t)}, \sin{(2\pi t)}\big)$. Now let's build a homotopy. Fortunately c itself is already a homotopy: we just check the three conditions $c(0,t) = b(0,0)$, $c(1,t) = b\big(\cos{(2\pi t)}, \sin{(2\pi t)}\big)$, and $c(s,0) = b(s\cos 0, s\sin 0) = b(s\cos 2\pi, s\sin 2\pi) = c(s,1)$.

4. Prove that the right side of Koszul formula (19.3.5) really does satisfy the conditions (19.3.4) (that is, tensorial in U and a derivation in V).

   Solution: Recall the formula $[fU, V] = f[U,V] - V(f)U$, which is easy to derive. From this we have for any vector field W that

$$\begin{aligned}\langle \nabla_{fU} V, W\rangle &= \tfrac{1}{2}\Big(fU\langle V,W\rangle - \langle V, [fU,W]\rangle - W\langle V, fU\rangle + \langle [W,V], fU\rangle \\ &\qquad + V\langle W, fU\rangle - \langle W, [V, fU]\rangle\Big)\\ &= \tfrac{1}{2}\Big(fU\langle V,W\rangle - \langle V, f[U,W]\rangle + W(f)\langle V,U\rangle - W(f)\langle V,U\rangle \\ &\qquad - fW\langle V,U\rangle + f\langle [W,V],U\rangle + V(f)\langle W,U\rangle + fV\langle W,U\rangle \\ &\qquad - V(f)\langle W,U\rangle - f\langle W,[V,U]\rangle\Big)\\ &= \tfrac{1}{2}\Big(fU\langle V,W\rangle - f\langle V,[U,W]\rangle - fW\langle V,U\rangle + f\langle [W,V],U\rangle \\ &\qquad + fV\langle W,U\rangle - f\langle W,[V,U]\rangle\Big)\\ &= f\langle \nabla_U V, W\rangle.\end{aligned}$$

   Since this is true for every W and the inner product is nondegenerate, we must actually have $\nabla_{fU} V = f\nabla_U V$ for any U and V.

The computation for $\nabla_U(fV)$ is similar. We have

$$\begin{aligned}
\langle \nabla_U(fV), W\rangle &= \tfrac{1}{2}\Big(U\langle fV, W\rangle - \langle fV, [U,W]\rangle - W\langle fV, U\rangle + \langle [W, fV], U\rangle \\
&\qquad + fV\langle W, U\rangle - \langle W, [fV, U]\rangle\Big) \\
&= \tfrac{1}{2}\Big(U(f)\langle V, W\rangle + fU\langle V, W\rangle - f\langle V, [U,W]\rangle - W(f)\langle V, U\rangle \\
&\qquad - fW\langle V, U\rangle + W(f)\langle V, U\rangle + f\langle [W,V], U\rangle + fV\langle W, U\rangle \\
&\qquad - f\langle W, [V,U]\rangle + U(f)\langle W, V\rangle\Big) \\
&= \tfrac{1}{2}\Big(2U(f)\langle V, W\rangle + fU\langle V, W\rangle - f\langle V, [U,W]\rangle - fW\langle V, U\rangle \\
&\qquad + f\langle [W,V], U\rangle + fV\langle W, U\rangle - f\langle W, [V,U]\rangle\Big) \\
&= U(f)\langle V, W\rangle + f\langle \nabla_U V, W\rangle.
\end{aligned}$$

Since this is true for all vector fields W, we must have $\nabla_U(fV) = U(f)V + f\nabla_U V$ as desired.

5. Suppose a surface in $\mathbb{R}^3$ is described by $z = x^2 - y^2$, so that it can be parametrized as $(u,v) \mapsto \big(u, v, u^2 - v^2\big)$. Compute the metric induced on the surface (u,v) by the metric on $\mathbb{R}^3$, as in the general formula in the middle of page 258 (between Examples 19.1.6 and 19.1.7). That is, find E, F, and G explicitly.

   Plug into formula (19.2.15) to find the sectional curvature K in this case; show that it's always negative.

   Solution: Here we have $f(u,v) = u$, $g(u,v) = v$, and $h(u,v) = u^2 - v^2$. From the cited formula, we have

$$\begin{aligned}
E &= f_u^2 + g_u^2 + h_u^2 = 1 + 4u^2 \\
F &= f_u f_v + g_u g_v + h_u h_v = -4uv \\
G &= f_v^2 + g_v^2 + h_v^2 = 1 + 4v^2.
\end{aligned}$$

   Plugging into the formula to end all formulas, we note that the nonzero terms are (since $E_v = G_u = 0$)

$$\begin{aligned}
K &= -\frac{1}{4(EG - F^2)^2}\Big(-4EGF_{uv} + 4F^2F_{uv} - FE_uG_v + 2GE_uF_v - 4FF_uF_v + 2EF_uG_v\Big) \\
&= -\frac{1}{4(1+4u^2+4v^2)^2}\Big(16(1+4u^2)(1+4v^2) + 64u^2v^2(-4) + 4uv(8u)(8v) \\
&\qquad + 2(1+4v^2)(8u)(-4u) - 4(-4uv)(-4v)(-4u) + 2(1+4u^2)(-4v)(8v)\Big) \\
&= -\frac{1}{4(1+4u^2+4v^2)^2}\Big(16(1+4u^2+4v^2) + 256u^2v^2 - 64u^2 - 256u^2v^2 \\
&\qquad + 256u^2v^2 - 64v^2 - 256u^2v^2\Big) \\
&= -\frac{4}{(1+4u^2+4v^2)^2}.
\end{aligned}$$

It certainly is negative.

6. For the metric
$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$
on the upper half-plane, find all the nonzero Christoffel symbols either from formulas (19.2.11) or (19.3.8), and verify that the geodesic equation $\frac{D}{dt}\frac{d\gamma}{dt} = 0$ from the middle of page 270 takes the form
$$\frac{d^2x}{dt^2} - \frac{2}{y}\frac{dx}{dt}\frac{dy}{dt} = 0, \qquad \frac{d^2y}{dt^2} + \frac{1}{y}\left(\frac{dx}{dt}\right)^2 - \frac{1}{y}\left(\frac{dy}{dt}\right)^2 = 0.$$
Check that $x(t) = a + (\tanh t)/b$ and $y(t) = (\operatorname{sech} t)/b$ are solutions for any constants a and b, thus showing that the geodesics are upper semicircles.

Solution: The metric components are $g_{11} = g_{22} = 1/y^2$ and $g_{12} = 0$. As a matrix it's given by $g = \begin{pmatrix} y^{-2} & 0 \\ 0 & y^{-2} \end{pmatrix}$. This is a diagonal matrix, so its inverse components are easy to compute: we have $g^{11} = g^{22} = y^2$ and $g^{12} = 0$. Plugging into formula (19.2.11) (with $u = x$ and $v = y$) we get
$$\Gamma^1_{11} = 0, \qquad \Gamma^2_{11} = -\frac{EE_y}{2y^{-4}} = \frac{1}{y}, \qquad \Gamma^1_{12} = \frac{GE_y}{2y^{-4}} = -\frac{1}{y}$$
$$\Gamma^2_{12} = 0, \qquad \Gamma^1_{22} = 0, \qquad \Gamma^2_{22} = \frac{EG_y}{2y^{-4}} = -\frac{1}{y}.$$

The geodesic equation is
$$\begin{aligned}
\frac{d^2x}{dt^2} + \Gamma^1_{11}\left(\frac{dx}{dt}\right)^2 + 2\Gamma^1_{12}\frac{dx}{dt}\frac{dy}{dt} + \Gamma^1_{22}\left(\frac{dy}{dt}\right)^2 &= 0 \\
\frac{d^2x}{dt^2} - \frac{2}{y}\frac{dx}{dt}\frac{dy}{dt} &= 0 \\
\frac{d^2y}{dt^2} + \Gamma^2_{11}\left(\frac{dx}{dt}\right)^2 + 2\Gamma^2_{12}\frac{dx}{dt}\frac{dy}{dt} + \Gamma^2_{22}\left(\frac{dy}{dt}\right)^2 &= 0 \\
\frac{d^2y}{dt^2} + \frac{1}{y}\left(\frac{dx}{dt}\right)^2 - \frac{1}{y}\left(\frac{dy}{dt}\right)^2 &= 0,
\end{aligned}$$
which is what we wanted to show.

Now let's just try plugging in $x = a + (\tanh t)/b$ and $y = (\operatorname{sech} t)/b$. For the x equation we obtain $-2\sinh t/(b\cosh^3 t) - 2/b\cosh t \cdot / \cosh^2 t(-2\operatorname{sech} t \tanh t) = 0$, and for the y equation we obtain
$$\frac{\sinh^2 t - 1}{b\cosh^3 t} + \frac{\cosh t}{b}\left(\frac{1}{\cosh^4 t} - \frac{\sinh^2 t}{\cosh^4 t}\right) = 0.$$
Hence the given formulas actually do satisfy the geodesic equations.