Math 70900 Homework #10
due Friday, December 8

Read "Introduction to Differential Geometry" through Chapter 19.

1. Let
$$\omega = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$
on $\mathbb{R}^3 \backslash \{(0,0,0)\}$. Show that ω is closed but not exact. Hint: to show it's not exact, integrate it over a parametrized 2-sphere and obtain a nonzero number.

2. (a) Suppose α is a closed k-form and β is an exact ℓ-form; show that $\alpha \wedge \beta$ is an exact $(k+\ell)$-form.

   (b) Now consider the cohomology quotient spaces $H^k(M)$, where we say that two closed forms α_1 and α_2 are equivalent, $\alpha_1 \equiv \alpha_2$, if $\alpha_1 - \alpha_2 = d\phi$ for some $(k-1)$-form ϕ. Show that if $\alpha_1 \equiv \alpha_2$ as closed k-forms and $\beta_1 \equiv \beta_2$ as closed ℓ-forms, then $\alpha_1 \wedge \beta_1 \equiv \alpha_2 \wedge \beta_2$ as closed $(k+\ell)$-forms. The product induced on the cohomology spaces by the wedge is called the *cup product on de Rham cohomology.*

3. A smooth closed curve $\gamma\colon [0,1] \to M$ (with $\gamma(0) = \gamma(1)$) is called *smoothly contractible* if there is a point $p \in M$ and a smooth map $H\colon [0,1] \times [0,1] \to M$ such that

   - $H(0,t) = p$ for all $t \in [0,1]$;
   - $H(1,t) = \gamma(t)$ for all $t \in [0,1]$;
   - $H(s,0) = H(s,1)$ for all $s \in [0,1]$.

   If γ is smoothly contractible, show that $\gamma = \partial H$. Conversely if γ is the boundary of a disc (that is, a map $c\colon [0,1] \times [0,1] \to M$ of the form $c(r,\theta) = b\big(r\cos(2\pi\theta), r\sin(2\pi\theta)\big)$ for some smooth $b\colon \mathbb{R}^2 \to M$), show that γ is smoothly contractible.

4. Prove that the right side of Koszul formula (19.3.5) really does satisfy the conditions (19.3.4) (that is, tensorial in U and a derivation in V).

5. Suppose a surface in $\mathbb{R}^3$ is described by $z = x^2 - y^2$, so that it can be parametrized as $(u,v) \mapsto \big(u, v, u^2 - v^2\big)$. Compute the metric induced on the surface (u,v) by the metric on $\mathbb{R}^3$, as in the general formula in the middle of page 258 (between Examples 19.1.6 and 19.1.7). That is, find E, F, and G explicitly.

   Plug into formula (19.2.15) to find the sectional curvature K in this case; show that it's always negative.

6. For the metric
$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

on the upper half-plane, find all the nonzero Christoffel symbols either from formulas (19.2.11) or (19.3.8), and verify that the geodesic equation $\frac{D}{dt}\frac{d\gamma}{dt} = 0$ from the middle of page 270 takes the form

$$\frac{d^2x}{dt^2} - \frac{2}{y}\frac{dx}{dt}\frac{dy}{dt} = 0, \qquad \frac{d^2y}{dt^2} + \frac{1}{y}\left(\frac{dx}{dt}\right)^2 - \frac{1}{y}\left(\frac{dy}{dt}\right)^2 = 0.$$

Check that $x(t) = a + (\tanh t)/b$ and $y(t) = (\operatorname{sech} t)/b$ are solutions for any constants a and b, thus showing that the geodesics are upper semicircles.