Math 6230 Homework #1
due Wednesday, September 6

Read "Introduction to Differential Geometry" up to Chapter 4.

1. Suppose you lived on a small sphere of radius R, so that spherical geometry was more natural than planar geometry. What formulas would children be taught in school instead of the Euclidean formulas $C = 2\pi r$ (for circumference of a circle) and $A = \pi r^2$ (for area of a circle)?

2. Recall that in the hyperbolic plane (the upper half-plane model $H = \{(x, y) : y > 0\}$) of hyperbolic geometry, "lines" are either semicircles centered on the x-axis or vertical lines, as shown in Figure 1.5.

   Show that for any two points (x_1, y_1) and (x_2, y_2) on the hyperbolic plane there is a unique hyperbolic line passing through the points; compute it explicitly.

3. In standard spherical coordinates on S^2, with $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$, and $z = \cos\theta$, show that $\sin^2\theta$ is a smooth function on the sphere (because it is the restriction of a smooth function on $\mathbb{R}^3$), but that $\sin^2\phi$ is not a smooth function on the sphere. (Hint: what does it look like near the north pole in (x, y) coordinates?)

4. Consider two possible bases for $\mathbb{R}^2$:

$$e_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \text{vs.} \qquad f_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad \text{and} \quad f_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

   (a) Find the transformation matrix P such that $f_i = \sum_{j=1}^{2} p_i^j e_j$, and the transformation matrix Q such that $e_i = \sum_{j=1}^{2} q_i^j f_j$.

   (b) How would you express the vector $v = 7e_1 - 3e_2$ in the $\{f\}$-basis?

5. A linear operator $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ is given in the standard basis $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ by

$$T = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -2 \\ -1 & 0 & -1 \end{pmatrix}.$$

   (a) If you think of the domain and range of T as *different* vector spaces, show how to change the bases of both to get T in reduced row echelon form. (Describe the new bases explicitly.)

   (b) If you think of the domain and range of T as the *same* vector space, show how to change the basis to get T in Jordan form.

6. Prove directly, imitating Proposition 3.3.1, that if $T \colon V \to V$ is a linear operator, then the number $\mathrm{Tr}(T^2) = \sum_{i=1}^{n} \sum_{j=1}^{n} T_i^j T_j^i$ does not depend on choice of basis.