Math 70900 Final Exam
Due Monday, December 8 by email

- Do all six questions.
- You may use only class notes (Introduction to Differential Geometry) and homework solutions.
- You may not consult with anyone except me.

1. Suppose $Q\colon \mathbb{R}^2 \to \mathbb{R}$ is a C^∞ function, and let M be the surface
$$M = \{(x, y, z) \in \mathbb{R}^3 \mid z = Q(x, y)\}.$$
   (a) Compute the induced Riemannian metric on M in coordinates.
   (b) At a critical point of Q, compute the sectional curvature.
   (c) Show that if Q has a local maximum or minimum, the sectional curvature must be positive.
   (d) Why does this prove that no compact surface of negative curvature can be embedded isometrically in $\mathbb{R}^3$?

2. Let M be a smooth manifold and α a smooth 1-form on M.
   (a) If M is two-dimensional with $d\alpha|_p \neq 0$ and $\alpha_p \neq 0$, show that there is a coordinate chart (ϕ, U) around p such that $\alpha = y\,dx$ in terms of the coordinates (x, y). (Hint: find a vector field X such that $\alpha(X) \equiv 0$, and straighten it.)
   (b) If M is three-dimensional and α is a contact form (see Homework #9 problem 4), then there is a coordinate chart (ϕ, U) such that $\alpha = dz + y\,dx$ in coordinates (x, y, z). (Hint: straighten the Reeb field to $\xi = \frac{\partial}{\partial w}$, and show that $\alpha = dw + f(u, v)\,du + g(u, v)\,dv$. Then apply part (a).)

3. Suppose ω is a k-form on a smooth manifold M. Consider a covariant derivative ∇ which is torsion-free and compatible with ω, i.e.,
$$\nabla_U V - \nabla_V U = [U, V], \qquad U\big(\omega(V_1, \ldots, V_k)\big) = \sum_{j=1}^{k} \omega(V_1, \ldots, \nabla_U V_j, \ldots, V_k).$$

(a) Show that if $k = 1$ or $k = 2$, and such a covariant derivative exists, then ω must be closed.

(b) If $\omega = dx$ on $\mathbb{R}^2$, is the covariant derivative uniquely determined by these two conditions?

(c) What if $\omega = dx \wedge dy$ on $\mathbb{R}^2$?

4. If G is a Lie group, let us say a 1-form α is *left-invariant* if $\alpha(V_1, \ldots, V_k)$ is a constant function on G for all left-invariant vector field $V_1, \ldots, V_k$.

   (a) Show that if ω is a left-invariant k-form, then so is $d\omega$.

   (b) Define the left-invariant cohomology to be the space of closed left-invariant forms modulo the space of exact left-invariant forms. Suppose G is three-dimensional with left-invariant fields E_1, E_2, and E_3, satisfying $[E_1, E_2] = \lambda_3 E_3$, $[E_2, E_3] = \lambda_1 E_1$, and $[E_3, E_1] = \lambda_2 E_2$. Show that the left-invariant cohomology of G is nontrivial if and only if at least one of the λ_i is zero.

5. Suppose α is a 1-form on $\mathbb{R}^3$ which is invariant under all rotations, i.e., $R^*\alpha = \alpha$ for every rotation $R\colon \mathbb{R}^3 \to \mathbb{R}^3$. Prove that α must be zero everywhere.

6. Let M be the manifold which is the 2-sphere with the north and south pole removed.

   (a) Show that M is smoothly homotopic to the circle (around the equator), i.e., find a smooth map $H\colon [0, 1] \times M \to M$ such that $H(0, x) = x$ for all $x \in M$ and $H(1, x) \in S^1 \times \{0\} \subset \mathbb{R}^3$ for all $x \in M$.

   (b) Describe an explicit 1-form on M which is closed but not exact, and prove both properties.