Math 70900 Nov. 8

Section 16.4: the pull back on forms

If $\eta : M \to N$ and we have vector fields X on M and Y on N, they are η-related if $\eta_\# X = Y \iff \eta_*(X_p) = Y_{\eta(p)}$.

It sometimes happens that given η and X there is a push-forward Y.

If $\eta: M \to N$ is smooth and ω is a k-form on N, then the pull-back $\eta^{\#}\omega$ is a k-form on M defined by

$$(\eta^{\#}\omega)_p(X_1, \cdots, X_k) = \omega_{\eta(p)}(\eta_*(X_1), \cdots, \eta_*(X_k))$$

$X_1, \cdots, X_k \in T_pM.$

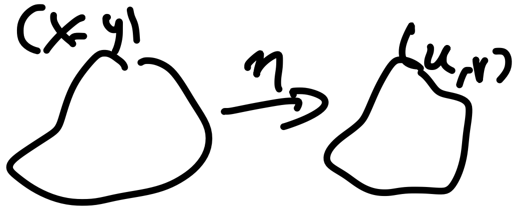
Alternate: $\eta^*\omega$

Example $\eta(x,y) = (2xy, x^2 - y^2)$

Compute $\eta^{\#}(x\,dy - y\,dx)$

$\eta: \mathbb{R}^2 \to \mathbb{R}^2$. $(u,v) = (2xy, x^2-y^2)$.

$\eta^{\#}(\omega) = \eta^{\#}(u\,dv - v\,du) = f(x,y)\,dx + g(x,y)\,dy.$



$$
\begin{aligned}
f(x,y) &= \eta^{\#}\omega\left(\frac{\partial}{\partial x}\right) \\
&= \omega_{\eta(x,y)}\left(\eta_{*}\frac{\partial}{\partial x}\right) \\
&= \omega\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right) \\
&= \omega\left(2y\frac{\partial}{\partial u} + 2x\frac{\partial}{\partial v}\right) \\
&= (u\,dv - v\,du)\left(2y\frac{\partial}{\partial u} + 2x\frac{\partial}{\partial v}\right) \\
&= 2xu - 2yv \\
&= 2x(2xy) - 2y(x^2 - y^2) \\
&= 2x^2y + 2y^3
\end{aligned}
$$

$g(x,y) = \eta^{\#}\omega\left(\frac{\partial}{\partial y}\right)$ is the same.

If $\eta : \mathbb{R}^2 \to \mathbb{R}^3$ then the space of 3-forms on $\mathbb{R}^3$ consists of $\omega = f(x,y,z)\,dx \wedge dy \wedge dz$. $\eta^{\#}\omega = 0$.

Generally this computation depends on the dimensions and k.

<u>Prop 16.4.3</u> $\eta^\#(\underline{\alpha\wedge\beta}) = \eta^\#\alpha \wedge \eta^\#\beta$

$d(\eta^\#\alpha) = \eta^\# d\alpha.$

Application: $\eta(x,y) = (u,v) = (2xy, x^2-y^2).$

$$\eta^\#(u\,dv - v\,du) = \eta^\#(u\,dv) - \eta^\#(v\,du)$$

$$= \eta^\# u \cdot \eta^\# dv - \eta^\# v \cdot \eta^\# du$$

For 0-forms f, $\eta^\# f = f\circ\eta$.

$$\eta^\#(u\,dv - v\,du) = (u\circ\eta)\cdot d(v\circ\eta) - (v\circ\eta)\,d(u\circ\eta)$$

$$= 2xy\,d(x^2-y^2) - (x^2-y^2)\,d(2xy)$$

$$= 2xy(2x\,dx - 2y\,dy) - (x^2-y^2)(2y\,dx + 2x\,dy)$$

$$= (4x^2y - 2x^2y + y^3)\,dx + (-4xy^2 - 2x^3 + 2xy^2)\,dy$$

Proof of Proposition

α j-form
β k-form

$$\eta^\#(\alpha\wedge\beta)_p(X_1,\dots,X_{j+k})$$

$$=(\alpha\wedge\beta)_p(\eta_* X_1,\dots,\eta_* X_{j+k})$$

$$=\sum_{\sigma\in S_{j+k}}\operatorname{sgn}(\sigma)\,\alpha(\eta_* X_{\sigma(1)},\dots,\eta_* X_{\sigma(j)})\cdot\beta(\eta_* X_{\sigma(j+1)},\dots,\eta_* X_{\sigma(j+k)})$$

$$=\sum\operatorname{sgn}(\sigma)\,\eta^\#\alpha(X_{\sigma(1)},\dots,X_{\sigma(j)})\cdot\eta^\#\beta(X_{\sigma(j+1)},\dots,X_{\sigma(j+k)})$$

$$=\eta^\#\alpha\wedge\eta^\#\beta(X_1,\dots,X_{j+k}).$$

(Really just $\eta^*(\alpha\wedge\beta)=\eta^*(\alpha)\wedge\eta^*(\beta)$ for forms in a vector space.)

$d(\eta^\# \alpha) = \eta^\# d\alpha$ is harder.

You can prove this in coordinates (see notes).

The reason it works is because d was defined to be coordinate-invariant.

If η is a diffeomorphism, then we can view this as just a coordinate change.

Still works even if η is not a diffeo because pullback makes sense no matter what.

Let's see the invariant proof.

($k=1$ for simplicity, α is a 1-form). Assume also that η is a diffeomorphism.

$$d(\eta^\# \alpha)(X,Y) = X(\eta^\# \alpha(Y)) - Y(\eta^\# \alpha(X)) - \underline{\eta^\# \alpha([X,Y])}.$$

To simplify this, note that

$$\cancel{\underbrace{\underset{\text{k-form on } M}{\eta^\# \alpha}(\underset{\text{v.f. on } M}{Y})}_{\text{function on } M} = \underbrace{\underset{\text{k-form on } N}{\alpha}(\underset{\text{v.f. on } N}{\eta_\# Y})}_{\text{function on } N}}$$

Not quite, actually

$$\eta^\# \alpha(Y) = \alpha(\eta_\# Y) \circ \eta$$

$$(\eta^\# \alpha)_p (Y_p) = \alpha(\eta_*(Y_p)) \quad \forall p \in M$$

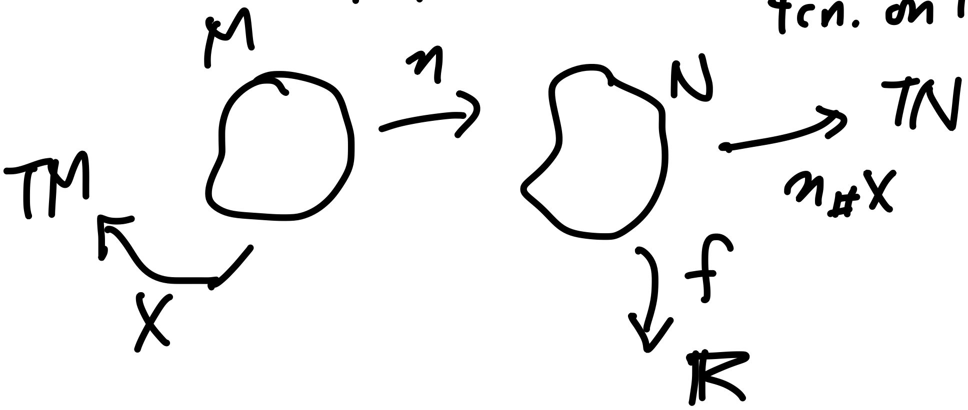
$$\forall q \in N \quad \alpha_q((\eta_\# Y)_q) = \alpha_q(\eta_*(Y_p)) \text{ where } q = \underline{\eta(p)}$$

$$= \alpha_{\eta(p)}(\eta_*(Y_p))$$

Returning we get

$$d(\eta^\# \alpha)(X,Y) = \underline{X(\alpha(\eta_\# Y)\circ\eta)}$$
$$- Y(\alpha(\eta_\# X)\circ\eta)$$
$$- \alpha([\eta_\# X, \eta_\# Y])\circ\eta$$

$$\underbrace{X(f\circ\eta)}_{\text{fcn. on } M} = \underbrace{(\eta_\# X)(f)\circ\eta}_{\text{fcn. on } M}$$



$$d(\eta^\# \alpha)(X,Y) = \eta_\# X(\alpha(\eta_\# Y))\circ\eta$$
$$- \eta_\# Y(\alpha(\eta_\# X))\circ\eta$$
$$- \alpha([\eta_\# X, \eta_\# Y])\circ\eta$$
$$= d\alpha(\eta_\# X, \eta_\# Y)\circ\eta$$
$$= \eta^\# d\alpha(X,Y). \qquad \square$$

§ 17: Integration and Stokes' Theorem

In vector calculus, we consider line integrals (over a curve), surface integrals in space, and volume integrals.

<u>Ex</u> Given a function f defined on $\mathbb{R}^2$ or $\mathbb{R}^3$ and a curve γ in that space, the line integral of f over γ is

$\int_\gamma f\, ds$, where ds is the "element of arc length."

In practice, we parameterize γ as $\gamma(t)$ for $a \le t \le b$.

Then $ds = |\gamma'(t)|\, dt$, and

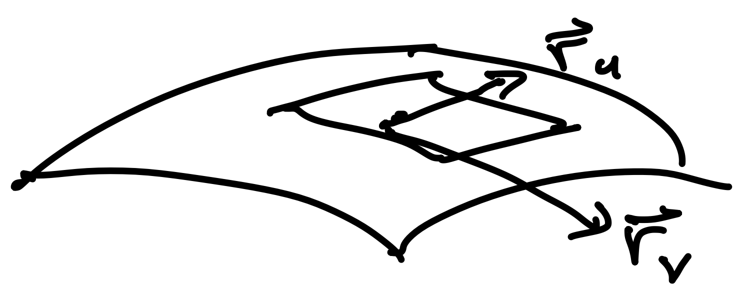
the line integral is

$$\int_\gamma f\,ds = \int_a^b f(\gamma(t))|\gamma'(t)|\,dt.$$

Surface integrals involve

$\int_\Sigma f\,dS$, where dS is the surface area element.

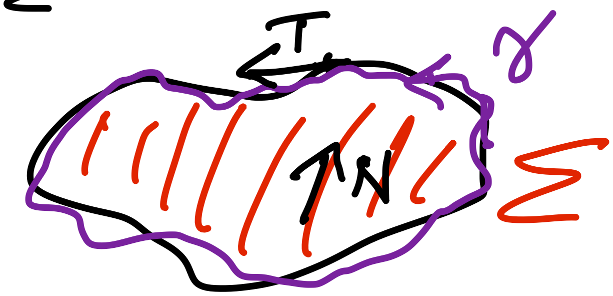
If Σ is parameterized by $\vec{r}(u,v)$, then $dS = \left|\frac{\partial \vec r}{\partial u} \times \frac{\partial \vec r}{\partial v}\right| du\,dv.$



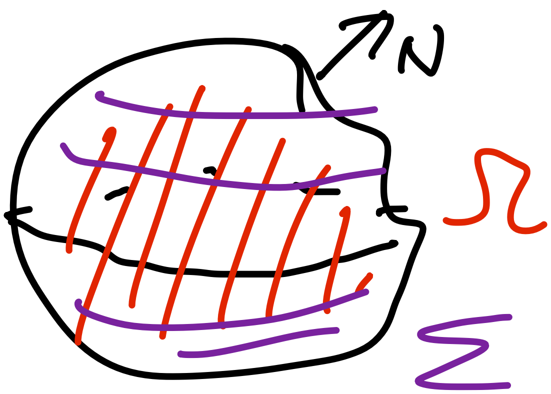
These appear in Stokes' Theorem and Divergence Theorem as

$$\int_\gamma X\cdot T\,ds = \iint_\Sigma \operatorname{curl} X\cdot N\,dS \qquad \text{Stokes'}$$

where



$$\iint_\Sigma X\cdot N\,dS = \iiint_\Omega \operatorname{div} X\,dV$$



When we compute $X\cdot N\,dS$ in coordinates using a parameterization, $\underline{X = p\hat{\imath} + q\hat{\jmath} + r\hat{k}}$, $x(u,v)$, $y(u,v)$, $z(u,v)$.

$$X\cdot N\,dS = X\cdot \frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\,|\vec{r}_u\times\vec{r}_v|\,du\,dv$$

$$= X\cdot \vec{r}_u\times\vec{r}_v\,\underline{du\,dv}$$

$$X \cdot \vec{r}_u \times \vec{r}_v = \begin{vmatrix} p & q & r \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$$

$$= p(y_u z_v - y_v z_u) + q(z_u x_v - x_u z_v) + r(x_u y_v - x_v y_u)$$

$$= p\, dy \wedge dz \left(\frac{\partial y}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial}{\partial z},\ \frac{\partial y}{\partial v}\frac{\partial}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial}{\partial z} \right)$$

$$= p\, dy \wedge dz \left(\iota_*\left(\frac{\partial}{\partial u}\right), \iota_*\left(\frac{\partial}{\partial v}\right) \right) + \cdots$$

where ι is the parameterization.

$$(p\, dy \wedge dz + q\, dz \wedge dx + r\, dx \wedge dy)\left(\iota_* \frac{\partial}{\partial u}, \iota_* \frac{\partial}{\partial v}\right) du \wedge dv$$

$$= \iota^{\#} \omega$$

where $\omega = p\, dy \wedge dz + \cdots$

The point is, all the line/surface integrals that actually show up in Green's, Stokes', Divergence Thm, are quite simple integrals in terms of a parameterization and a k-form and a pull-back.

We want to define an invariant notion of integration of a k-form on a k-dim. object in M.