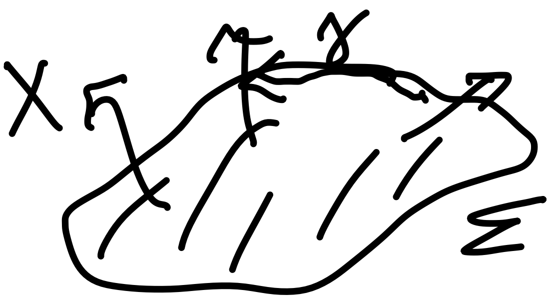
Differential Geometry, Nov. 10

Plan: Section 17, Integration and Stokes' Theorem

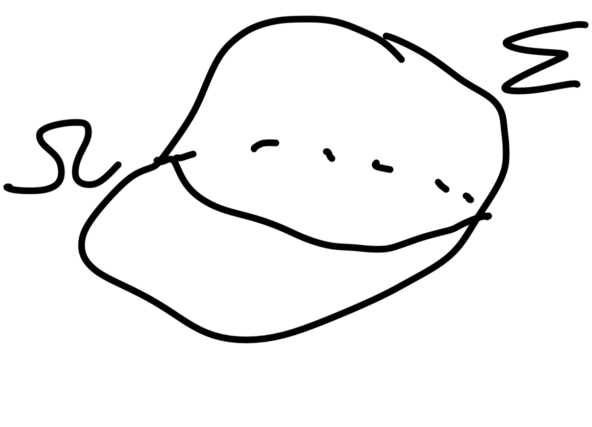
In vector calculus we have

Stokes' Thm



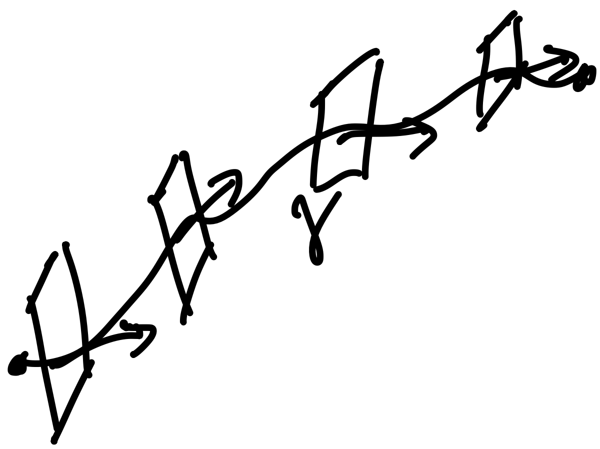
$$\oint_\gamma X\cdot T\,ds = \iint_\Sigma \operatorname{curl} X\cdot N\,dS$$

Divergence Thm



$$\oiint_\Sigma X\cdot N\,dS = \iiint_\Omega (\operatorname{div} X)\,dV$$

Easy case: integrating a 1-form ω over a curve in the manifold M.



We want to add up the action of ω on the tangent vector $\gamma'(t)$ for $t \in [a,b]$.

And we want the answer to be independent of choice of parameterization.

Start with $\gamma : [a,b] \to M$, and pull back ω by $\gamma^{\#}$ to get a 1-form on $[a,b]$.

$$\int_\gamma \omega := \int_a^b \gamma^{\#}\omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))\,dt.$$

<u>Example</u> $\omega = y\,dx - x\,dy + dz$

on $\mathbb{R}^3$, $\gamma(t) = (\cos t, \sin t, t)$.

on $[0, \pi]$.

$$\int_\gamma \omega = \int_0^\pi \omega_{\gamma(t)}(\gamma'(t))\,dt$$

$$= \int_0^\pi \sin t \, \underline{dx(\gamma'(t))} - \cos t \, dy(\gamma'(t)) + dz(\gamma'(t))\,dt$$

$$\gamma'(t) = -\sin t \frac{\partial}{\partial x} + \cos t \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

$dx(\gamma'(t)) = -\sin t$, $dy(\gamma'(t)) = \cos t$, $dz(\gamma'(t)) = 1$.

$$\underline{dx(\gamma'(t))} = \underline{d(x(\gamma(t)))} = d(\cos t) = -\sin t\,dt$$

$$= \frac{\partial}{\partial t}(\cos t)\,dt$$

$$\int_0^\pi -\sin^2 t - \cos^2 t + 1 \; dt = 0$$

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))\,dt.$$

Prove Prop 17.13: this is independent of choice of parameterization, except that it does depend on direction.

<u>Proof</u> Suppose $h:[c,d]\to[a,b]$ is strictly increasing with $h'(s)>0\ \forall s\in[c,d]$.

$\tilde\gamma=\gamma\circ h$, check that

$$\int_{\tilde\gamma}\omega=\int_\gamma\omega.$$

$$\int_{\tilde\gamma}\omega=\int_c^d\omega_{\tilde\gamma(s)}(\tilde\gamma'(s))\,ds$$

$$\tilde\gamma'(s)=\frac{d}{ds}\gamma(h(s))=\gamma'(h(s))h'(s)$$

$$\int_{\tilde\gamma}\omega=\int_c^d\omega_{\gamma(h(s))}(\gamma'(h(s))h'(s))\,ds$$

$$= \int_c^d \omega_{\gamma(h(s))}\left(\gamma'(h(s))\right) \underline{h'(s)\,ds}$$

$t = h(s)$ $$= \int_a^b \omega_{\gamma(t)}\left(\gamma'(t)\right) dt = \int_\gamma \omega. \quad \square$$

If h is decreasing, the sign flips.

Ex $\gamma(t) = (t, 0)$ and $\omega = x\,dx - y\,dy$.
$t \in [0,1]$. $h(t) = 1 - t$ on $[0,1]$.

$\tilde{\gamma}(t) = (1-t, 0)$

$d(x \circ \tilde{\gamma}) = -dt$

$$\int_\gamma \omega = \int_0^1 t\,dt = \frac{1}{2}$$

$$\int_{\tilde{\gamma}} \omega = \int_0^1 (1-t)(-dt) = -\frac{1}{2}.$$

So we need some orientation specified, not merely the point set.

Stokes' Theorem in one-dimension

Thm If $\omega = df$ then for any curve $\gamma : [a,b] \to M$ we have $\int_\gamma df = f(\gamma(b)) - f(\gamma(a))$.

Proof
$$\int_\gamma df = \int_a^b df(\gamma'(t))\,dt$$
$$= \int_a^b \frac{d}{dt}(f(\gamma(t)))\,dt$$
$$= \underline{f(\gamma(b)) - f(\gamma(a))}. \quad \square$$

Def The boundary of a curve $\gamma : [a,b] \to M$ is defined to be a "formal sum" of its endpoints, $\partial\gamma = \gamma(b) - \gamma(a)$.

The formal sum is defined by what it does to 0-forms.
$f : M \to \mathbb{R}$

$$\int_{\partial\gamma} f := f(\gamma(b)) - f(\gamma(a)).$$

$$\boxed{\int_\gamma df = \int_{\partial\gamma} f}$$

int of 1-form over a 1-d object

int of a 0-form over a 0-d object

If γ is a closed curve, so that $\gamma(b) = \gamma(a)$, then

$$\partial\gamma = \gamma(b) - \gamma(a) = 0.$$

We mean $\partial\gamma = 0$ in the sense that it produces zero when any 0-form is applied to it.

Thus if γ is closed and $\omega = df$ for some f, then $\int_\gamma \omega = 0$.
(i.e., ω is exact).

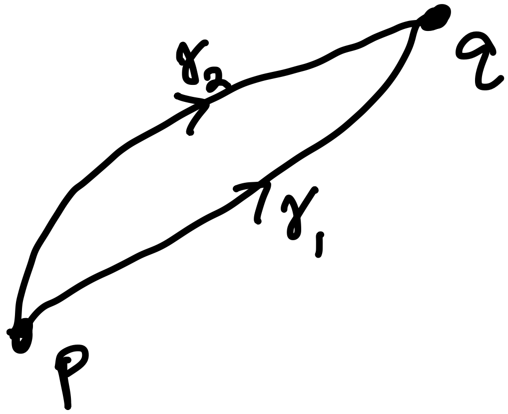
Generally we can define formal sums of curves $\gamma_1, \dots, \gamma_m$ as linear combinations

$$\sum_{i=1}^{m} a_i \gamma_i \quad \text{for } a_i \in \mathbb{Z},$$

by the action of 1-forms:

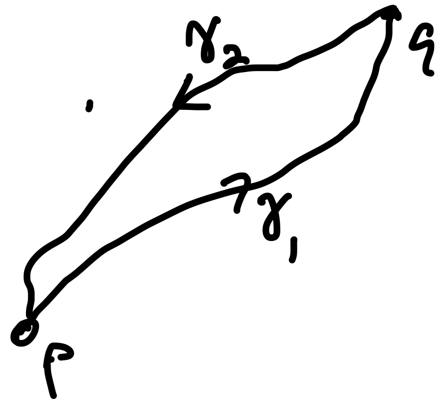
$$\int_{\sum_i a_i \gamma_i} \omega := \sum_i a_i \int_{\gamma_i} \omega .$$

Two formal sums are equal if they give the same number for any 1-form integrated over them.



$$\int_{\gamma_1} df = \int_{\gamma_2} df$$

for any function f and any two curves γ_1, γ_2 with the same endpoints.



$$\gamma = \gamma_1 - \gamma_2$$
$$\int_\gamma df = 0.$$

<u>Def</u> A <u>1-cube</u> is a curve $\gamma : [0,1] \to M$, and a <u>1-chain</u> is a formal sum of 1-cubes with integer coefficients.

1-chains are the natural things to integrate 1-forms.

0-cube is a point in M, and a 0-chain is a formal sum of 0-cubes.

<u>Def</u> The <u>boundary</u> of a 1-chain $\gamma = \sum a_i \gamma_i$ is $\partial\gamma = \sum a_i \partial\gamma_i$

$$= \sum a_i (\gamma_i(1) - \gamma_i(0))$$

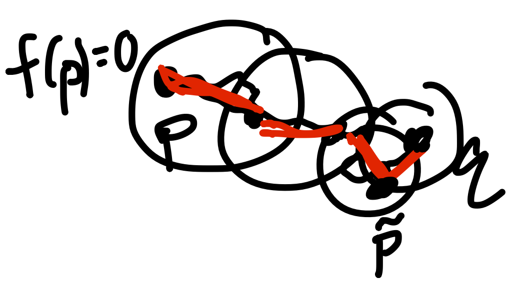
Theorem A 1-form ω is exact, $\omega = df$ for some $f: M \to \mathbb{R}$, iff $\int_\gamma \omega = 0$ for every 1-chain γ which is closed, i.e., $\partial\gamma = 0$.

Proof If $\omega = df$ then we already saw $\int_\gamma df = \int_{\partial\gamma} f = 0$.

We need to prove the converse.

We want to construct f.

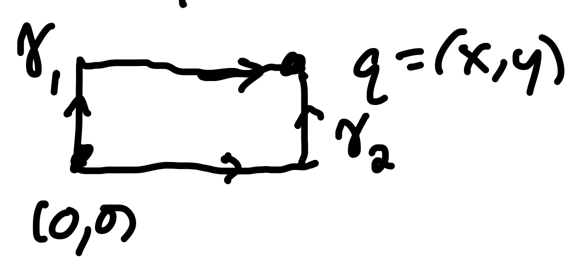
Pick a point p, and for any other q which is connected to p by some continuous path $\hat{\gamma}$, smooth out the path to a 1-chain γ, and define $f(q) = \int_\gamma \omega$.



By assumption if γ_1 and γ_2 are two 1-chains with
$$\partial\gamma_1 = q-p \text{ and } \partial\gamma_2 = q-p,$$
then $\gamma_2 - \gamma_1$ is a closed 1-chain, and since $\int_{\gamma_2-\gamma_1}\omega = 0$ we see that $f(q)$ is independent of choice of curve.

We want to verify $df = \omega$. Let's do it in coordinates. Assume WLOG that p and q are in the same chart.

$f(q) = \int_\gamma \omega$. We can assume $p = (0,0,\cdots,0)$ and $q = (x^1,\cdots,x^n)$.



Want to show $\frac{\partial f}{\partial x^i}(x^1, \dots, x^n) = \omega_i(x^1, \dots, x^n)$

for each i.

Then define curves

$\gamma_1(t) = (x^1 t, 0, \dots, 0)$ on $[0,1]$

$\gamma_2(t) = (x^1, t x^2, \dots, 0)$ on $[0,1]$

$\vdots$

$\gamma_{n-1}(t) = (x^1, x^2, \dots, 0, \dots, t x^n)$ on $[0,1]$

$\uparrow$ i^{th} place

$\gamma_n(t) = (x^1, x^2, \dots, t x^i, \dots, x^n)$ on $[0,1]$.

$\gamma_n'(t) = x^i \frac{\partial}{\partial x^i}$

$$\partial\left(\sum_{j=1}^{n} \gamma_j\right) = q - p$$

$$f(q) = f(x^1, \dots, x^n) = f(x^1, \dots, 0, \dots, x^n)$$

$$\int_{\gamma_n} \omega_{\gamma_n(t)}(\gamma_n'(t))\, dt$$

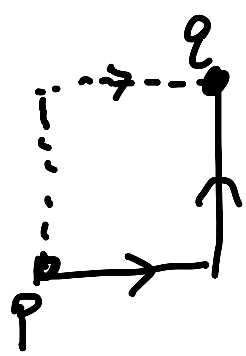
$$\frac{\partial f}{\partial x^i}(x^1, \dots, x^n) = \frac{\partial}{\partial x^i} \int_0^1 \omega_i(x^1, \dots, t x^i, \dots, x^n)\, x^i\, dt$$

$$= \frac{\partial}{\partial x^i} \int_0^{x^i} \omega_i(x^1, \cdots, s, \cdots, x^n)\, ds$$

$$= \omega_i(x^1, \cdots, x^i, \cdots, x^n).$$

True for all i, thus $df = \omega$. $\square$

Recall we did basically the same thing when proving the Poincaré Lemma on $\mathbb{R}^2$.



If $d\omega = 0$ then $\omega = df$.

$$f(x,y) = \int_0^x \omega_1(s,0)\, ds + \int_0^y \omega_2(x,t)\, dt.$$

To prove $df = \omega$ we have to differentiate w.r.t. x

$$\frac{\partial f}{\partial x}(x,y) = \omega_1(x,0) + \int_0^y \frac{\partial \omega_2}{\partial x}(x,t)\, dt$$

$$\frac{\partial f}{\partial y}(x,y) = \omega_2(x,y).$$

$\int_\gamma \omega = 0 \quad \forall\ \gamma$ with $\partial\gamma = 0$

is a stronger statement than

$d\omega = 0$.

The first implies $\omega = df$ for some f on <u>any</u> manifold M.

The second only implies it on simply connected spaces.

Example 17.1.9

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

"$\omega = d\theta$"

Prove that ω is not exact.

If it were exact then

$\int_\gamma \omega = 0 \quad \forall$ closed γ.

$\gamma(t) = (\cos t, \sin t)$. on $[0, 2\pi]$

$dx = d(\cos t) = -\sin t\, dt$

$dy = d(\sin t) = \cos t\, dt$

$$\int_\gamma \omega = \int_0^{2\pi} \frac{(-\sin t)(-\sin t\, dt) + \cos^2 t\, dt}{\cos^2 t + \sin^2 t}$$

$$= \int_0^{2\pi} dt = 2\pi \neq 0.$$