SINGULARITIES OF THE EXPONENTIAL MAP ON THE VOLUME-PRESERVING DIFFEOMORPHISM GROUP

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1. Introduction

Let M be a compact, oriented Riemannian manifold of dimension n, possibly with smooth boundary ∂M. Let $\mathcal{D}^s_\mu(M)$ denote the group of all diffeomorphisms of Sobolev class H^s preserving the volume form μ on M. If $s > n/2+1$ then $\mathcal{D}^s_\mu(M)$ becomes an infinite dimensional Hilbert manifold whose tangent space at a point η consists of H^s sections X of the pull-back bundle η^*TM for which the corresponding vector field $U = X\circ\eta^{-1}$ on M is divergence free and tangent to ∂M. Using right translations, the L^2 inner product on vector fields,

$$\langle\langle U, W\rangle\rangle_{L^2} = \int_M \langle U(x), W(x)\rangle\, d\mu(x), \qquad U, W \in T_{\mathrm{id}}\mathcal{D}^s_\mu(M) \tag{1.1}$$

defines a right-invariant metric on the group. Our main reference for basic facts regarding $\mathcal{D}^s_\mu(M)$ is the paper of Ebin and Marsden [EMa].

A strong motivation to study the geometry of diffeomorphism groups comes from hydrodynamics. In a seminal paper, Arnold [A] related motions of a perfect fluid in M to geodesics in $\mathcal{D}_\mu(M) = \cap_s \mathcal{D}^s_\mu(M)$. He observed that a curve $\eta(t)$ is a geodesic of the L^2 metric (1.1) starting from the identity id in the direction V_o if and only if the time dependent vector field $V = \dot\eta\circ\eta^{-1}$ on M solves the Euler equation of hydrodynamics:

$$\begin{gathered} \partial_t V + \nabla_V V = -\operatorname{grad} p, \\ \operatorname{div} V = 0, \\ V(0) = V_o, \qquad \langle V, \nu\rangle = 0 \text{ on } \partial M, \end{gathered} \tag{1.2}$$

where ∇ denotes the covariant derivative, p is the pressure function and ν is the normal to ∂M.

Soon after, Ebin and Marsden [EMa] discovered that there is a technical advantage in rewriting the Euler equation in this way. Their result was that the corresponding geodesic equation on the group $\mathcal{D}^s_\mu(M)$ is in fact a smooth ODE and can therefore be solved uniquely for small values of t using a standard Picard iteration argument. Furthermore, since the solutions depend smoothly on initial data, it follows that the L^2 metric has a smooth exponential map

$$\exp_{\mathrm{id}} : T_{\mathrm{id}}\mathcal{D}^s_\mu \to \mathcal{D}^s_\mu(M)$$

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defined, for small t, by

$$\exp_{\mathrm{id}}(tV_o) = \eta(t),$$

where η is the unique geodesic from the identity with initial velocity V_o. By the inverse function theorem, this map is a local diffeomorphism from $T_{\mathrm{id}}\mathcal{D}^s_\mu$ to a neighborhood of the identity in $\mathcal{D}^s_\mu(M)$.

In [A] Arnold also suggested that this approach could be used to study stability of fluid motions through the equation of geodesic deviation. This led to extensive calculations of the curvature of the diffeomorphism group and to the search for conjugate points in $\mathcal{D}^s_\mu(M)$. Many of these and related results together with references were assembled in the book by Arnold and Khesin [AK].

Recall that a point $\gamma(t)$ is said to be conjugate to the point $\gamma(0)$ if the derivative $d\exp_{\gamma(0)}(t\dot\gamma(0))$, identified with a map from the tangent space at $\gamma(0)$ to the one at $\gamma(t)$, is singular. In contrast with finite dimensional geometry, two types of singularities can occur in a general Hilbert manifold. Following Grossman [G], we say that a conjugate point $\gamma(t)$ is *monoconjugate* (respectively *epiconjugate*) to $\gamma(0)$ if $d\exp_{\gamma(0)}(t\dot\gamma(0))$ fails to be one-to-one (respectively onto). In finite dimensions the two types clearly coincide.

For general Hilbert manifolds many different conjugacies are possible. A simple example of the unit sphere in the space ℓ^2 of square-summable sequences with the induced metric shows that conjugate points may have infinite order. Another example, due to Grossman, of an ellipsoid in the same space shows that finite geodesic segments may contain infinitely many conjugate points of either type.

In the case of diffeomorphism groups, if the underlying manifold M has positive curvature and enough symmetry, then conjugate points can be located along those geodesics which are contained in the isometry subgroup of $\mathcal{D}^s_\mu(M)$; see Misiołek [M1]. It is also known, though more difficult to prove, that conjugate points exist in $\mathcal{D}^s_\mu(M)$ even if M is flat. See Misiołek [M2] for the case when M is a flat torus $\mathbb{T}^2$ and Shnirelman [Sh] for the ball in $\mathbb{R}^3$. It is therefore of interest to investigate the nature of conjugate points in $\mathcal{D}^s_\mu(M)$. We will prove the following result.

Theorem 1. *Let M^2 be a compact two-dimensional manifold without boundary. Then the exponential map of the L^2 metric on $\mathcal{D}^s_\mu(M^2)$ is a nonlinear Fredholm map of index zero.*

Recall that a bounded linear operator L between Banach spaces is said to be *Fredholm* if it has closed range and both its kernel and cokernel are finite dimensional. L is said to be *semi-Fredholm* if it has closed range and at least one of the other two conditions holds. The *index* of a semi-Fredholm operator is defined as

$$\operatorname{ind} L = \dim\ker L - \dim\operatorname{coker} L.$$

Semi-Fredholm operators form an open set in the space of all bounded linear operators and the index is a continuous function on this set into $\mathbb{Z}\cup\{\pm\infty\}$; see for example [K]. A smooth map f between Banach manifolds is called a *Fredholm map* if its Frechet derivative $df(p)$ is a Fredholm operator for each p. If the domain of f is connected then the index of the operator $df(p)$ is independent of p and by definition is the *index* of f. Fredholm maps were introduced by Smale [Sm].

Observe that as a corollary to Theorem 1 we find that the monoconjugate and epiconjugate points in $\mathcal{D}^s_\mu(M^2)$ coincide if M has no boundary. Furthermore, they are isolated and of finite multiplicity along finite geodesic segments. Also, by the infinite dimensional version of Sard's Theorem (see [Sm]) the set of all such points must be of 1st Baire category in $\mathcal{D}^s_\mu(M^2)$. Thus the structure of singularities of $\exp_{\mathrm{id}}$ essentially looks like that of a finite-dimensional manifold. A similar result is known to hold for the exponential map on the free loop space of a compact manifold M with its natural Sobolev H^1 metric, see [M3]. The above Theorem was announced for the flat torus $\mathbb{T}^2$ and conjectured for other manifolds in [EMi].

The techniques we use to prove Theorem 1 fail if the surface M has a nonempty boundary, and we do not know if the theorem is true in this case. The topology of $\mathcal{D}_\mu(M)$ is substantially more complicated when M has a boundary than when it does not. For example, Shnirelman [Sh] gave an example of an area-preserving diffeomorphism ξ on the disc, smooth on the interior and continuous up to the boundary, such that ξ cannot be joined to the identity by a curve of finite length in the L^2 norm (in particular, ξ is not in the image of $\exp_{\mathrm{id}}$). If Fredholmness fails on a manifold with boundary, it could be related to such examples.

We can however prove a weaker result valid for any surface, using the weak topology on each $T_{\mathrm{id}}\mathcal{D}_\mu$ generated by the L^2 norm, rather than the H^s norm for $s > n/2 + 1$. We denote the closure of $T_{\mathrm{id}}\mathcal{D}_\mu$ in this norm by $T_{\mathrm{id}}\mathcal{D}^0_\mu$. Since $\exp_{\mathrm{id}}$ is not necessarily defined on $T_{\mathrm{id}}\mathcal{D}^0_\mu$, we only have a result about the differential.

Theorem 2. *Suppose M is any compact surface, possibly with boundary. For any $V_o \in T_{id}\mathcal{D}^s_\mu(M)$ with $s > 2$, the differential $d\exp_{id}(tV_o)$ extends to a bounded operator from $T_{id}\mathcal{D}^0_\mu(M)$ to $T_{\exp(tV_o)}\mathcal{D}^0_\mu(M)$ which is Fredholm of index zero in the L^2 topology.*

Notice that this theorem tells us about the monoconjugate points in the H^s norm as well, since any point which is monoconjugate in the H^s sense is also monoconjugate in the L^2 sense. Thus for example, monoconjugate points cannot cluster on $\mathcal{D}^s_\mu(M^2)$ in a finite interval.

Combining Theorems 1 and 2, we can say that if $\eta(t) = \exp(tV_o)$ is not monoconjugate to id along the H^s geodesic η, then $d\exp_{\mathrm{id}}(tV_o)\colon T_{\mathrm{id}}\mathcal{D}^0_\mu \to T_{\eta(t)}\mathcal{D}^0_\mu$ is surjective, but the restricted map $d\exp_{\mathrm{id}}(tV_o)\colon T_{\mathrm{id}}\mathcal{D}^s_\mu \to T_{\eta(t)}\mathcal{D}^s_\mu$ may not be surjective. In other words, the open question is whether $d\exp_{\mathrm{id}}(tV_o)$ smooths out vector fields near the boundary or not.

In three dimensions, the situation changes dramatically: both Theorems 1 and 2 fail in this case. In particular, we prove in Section 4 that along the geodesic corresponding to uniform rotational flow of a flat solid torus $D^2 \times S^1$, there are epiconjugate points that are not monoconjugate points. We find epiconjugate points by first computing the monoconjugate points, using a basis of vector fields which diagonalizes the Jacobi equation. The set of monoconjugate points turns out not to be discrete, and in the course of the proof we find that the cluster points of this set are epiconjugate points.

In particular, if V_o is the initial velocity vector for uniform unit-speed rotational flow, then the cut point of the geodesic at $\exp_{\mathrm{id}}(\pi V_o)$ is an epiconjugate point but not a monoconjugate point, and thus the range of $d\exp_{\mathrm{id}}(\pi V_o)$ cannot be closed.

2. Background and preparation

It is useful to consider $\mathcal{D}^s_\mu(M)$ as a Riemannian submanifold of the group $\mathcal{D}^s(M)$ of all H^s diffeomorphisms equipped with the same L^2 metric. The action of $\mathcal{D}^s_\mu(M)$ on $\mathcal{D}^s(M)$ given by composition on the right is an isometry of (1.1) and combined with the Weyl decomposition gives an L^2 orthogonal splitting of each tangent space

$$T_\eta\mathcal{D}^s = T_\eta\mathcal{D}^s_\mu \oplus \operatorname{grad} H^{s+1}(M)\circ\eta.$$

The projections onto the first and second summands above depend smoothly on the base point η and will be denoted by P_η and Q_η respectively, or simply P and Q if $\eta = \mathrm{id}$. The L^2 metric induces a smooth invariant Levi-Civita connection $\tilde{\nabla}$ on $\mathcal{D}^s_\mu(M)$ whose curvature tensor $\tilde{R}$ is also invariant with respect to right multiplication by $\mathcal{D}^s_\mu(M)$. We refer to [EMa] for the proofs of these facts.

Let V_o be any vector in $T_{\mathrm{id}}\mathcal{D}^s_\mu$ and let η be the geodesic of (1.1) starting from the identity with initial velocity V_o. If M is two-dimensional, the Cauchy problem for the Euler equation is globally well-posed, and it follows that the geodesic can be extended indefinitely. (In three dimensions this is a major unsolved problem.) In order to study Fredholmness of $\exp_{\mathrm{id}}$, it will be convenient to express its derivative at tV_o in terms of solutions to the Jacobi equation

$$\tilde{\nabla}_{\dot\eta}\tilde{\nabla}_{\dot\eta}J + \tilde{R}(J,\dot\eta)\dot\eta = 0 \tag{2.3}$$

along $\eta(t) = \exp_{\mathrm{id}}(tV_o)$ with initial conditions

$$J(0) = 0, \quad \tilde{\nabla}_{V_o}J(0) = W_o. \tag{2.4}$$

Existence and uniqueness of (global in time) Jacobi fields solving (2.3)–(2.4) follows at once from the fact that $\tilde{R}$ is a smooth multi-linear operator and (2.3) is a linear equation. See Misiołek [M1] for details.

If J is the Jacobi field along η with the initial conditions (2.4), then

$$d\exp_{\mathrm{id}}(tV_o)tW_o = J(t). \tag{2.5}$$

Furthermore, the derivative of the exponential map at tV_o forms a family of linear operators $F(t)\colon T_{\mathrm{id}}\mathcal{D}^s_\mu \to T_{\eta(t)}\mathcal{D}^s_\mu$ which are bounded for each fixed t and which depend smoothly on t, and are given by $F(t)(W_o) = J(t)$.

We will prove that this operator $F(t)$ is Fredholm by using the decomposition of the Jacobi equation, first noticed by Rouchon [R] and exploited by Preston [P] to study Lagrangian stability of steady flows. First we note that if $X(t)$ is any time-dependent vector field along a curve $\eta(t)$, i.e. with $X(t)\in T_{\eta(t)}\mathcal{D}^s_\mu$, then we can right-translate X back to the identity to get a time-dependent vector field $Y(t)\in T_{\mathrm{id}}\mathcal{D}^s_\mu$, given by

$$Y(t) \equiv dR^{-1}_{\eta(t)}X(t) = X(t)\circ\eta^{-1}(t). \tag{2.6}$$

We can compute the covariant derivative of X along η using the formula

$$\tilde{\nabla}_{\dot\eta(t)}X(t) = \left(\frac{\partial Y}{\partial t} + P\big(\nabla_{V(t)}Y(t)\big)\right)\circ\eta(t), \tag{2.7}$$

where $V(t)$ is the Eulerian velocity field, defined by

$$\frac{\partial \eta}{\partial t} = V(t)\circ\eta(t).$$

From equation (2.7) and the geodesic equation $\tilde{\nabla}_{\dot\eta}\dot\eta = 0$, one derives the Euler equation of ideal, incompressible flow

$$\frac{\partial V}{\partial t} + P\big(\nabla_{V(t)}V(t)\big) = 0. \tag{2.8}$$

Using formula (2.7), the Euler equation (2.8), and the definition of the Riemann curvature operator, it is not difficult to verify that the Jacobi equation (2.3) is equivalent to the two equations

$$\frac{\partial Z}{\partial t} + P\big(\nabla_{V(t)}Z(t) + \nabla_{Z(t)}V(t)\big) = 0, \tag{2.9}$$

$$\frac{\partial Y}{\partial t} + [V(t), Y(t)] = Z(t), \tag{2.10}$$

where $Y(t) = J(t)\circ\eta^{-1}(t)$.

The reason this decoupling works is because the geodesic equation also decouples, into the Euler equation of an ideal fluid and the flow equation; both decouplings are due to the right-invariance of the metric on $\mathcal{D}^s_\mu(M)$. The same thing happens for any Lie group with a right-invariant Riemannian metric, and there is a corresponding result for the left-invariant case. The equation (2.9) is the linearized Euler equation, extensively studied in questions of linear stability of a steady incompressible flow, while the equation (2.10) is the linearization of the flow equation.

For $\sigma \geq 0$, let $T_{\mathrm{id}}\mathcal{D}^\sigma_\mu(M)$ denote the closure of the space of smooth, divergence-free vector fields, tangent to the boundary of M, in the H^σ norm. By the Weyl decomposition, this is a closed subspace of the space of all H^σ vector fields on M (see [EMa]). For $\sigma > n/2+1$, this coincides with the actual tangent space to $\mathcal{D}^\sigma_\mu(M)$. For smaller σ, $\mathcal{D}^\sigma_\mu(M)$ is not necessarily a smooth manifold.

It is helpful to observe the following. If $V \in T_{\mathrm{id}}\mathcal{D}^s_\mu(M)$, with $s > n/2+1$, then V is of class C^1, by the Sobolev embedding theorem. If we denote by $\mathcal{L}_V$ the (unbounded) operator from $T_{\mathrm{id}}\mathcal{D}^\sigma_\mu$ to itself given by

$$\mathcal{L}_V : Y \mapsto [V, Y], \tag{2.11}$$

for divergence-free vector fields V and Y tangent to ∂M, then the formal L^2-adjoint of $\mathcal{L}_V$ in $T_{\mathrm{id}}\mathcal{D}^0_\mu$ is easily computed to be

$$\mathcal{L}_V{}^\star(Z) = -P\big((\iota_V dZ^\flat)^\sharp\big) = -P(\nabla_V Z + \nabla_Z V) + K_V(Z), \tag{2.12}$$

where the operator K_V is defined as

$$K_V(Z) = P\big((\iota_Z dV^\flat)^\sharp\big). \tag{2.13}$$

(Here $\sharp$ denotes the operator of raising indices using the metric on M, to get a vector field from a 1-form, and $\flat$ denotes its inverse.)

If $s \geq \sigma + 1$, then the operator K_V is continuous on $T_{\mathrm{id}}\mathcal{D}^\sigma_\mu$, since $dV^\flat$ is of class H^{s-1} and P is continuous. It is anti-self-adjoint in $T_{\mathrm{id}}\mathcal{D}^0_\mu$ because $dV^\flat$ is antisymmetric.

Now, using equations (2.9) and (2.10), we can write the factorization of the Jacobi equation as

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_{V(t)}{}^\star + K_{V(t)}\right)\left(\frac{\partial}{\partial t} + \mathcal{L}_{V(t)}\right)Y(t) = 0. \tag{2.14}$$

Because the Jacobi equation is self-adjoint, we can factor it in another way as well.

Lemma 3. *The equation* (2.14) *is equivalent to the equation*

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_{V(t)}{}^\star\right)\left(\frac{\partial}{\partial t} + \mathcal{L}_{V(t)} + K_{V(t)}\right)Y(t) = 0. \tag{2.15}$$

Proof. This is a consequence of self-adjointness, but we can also verify the formula directly by proving that

$$\frac{d}{dt}K_{V(t)} = \mathcal{L}_{V(t)}{}^\star K_{V(t)} + K_{V(t)}\mathcal{L}_{V(t)}, \tag{2.16}$$

this formula arising from setting the difference of the operators in (2.14) and (2.15) equal to zero. Here the time derivative is taken in the operator sense.

In standard Lie group notation, the group adjoint operator is $\mathrm{Ad}_\eta = dR_{\eta^{-1}} dL_\eta$. Since on the diffeomorphism group, $dR_{\eta^{-1}}(X) = X \circ \eta^{-1}$ and $dL_\eta(X) = D\eta(X)$, the group adjoint is the push-forward operation of a diffeomorphism η on a vector field, given by

$$\mathrm{Ad}_\eta(X) = \eta_*(X) = D\eta \circ X \circ \eta^{-1}. \tag{2.17}$$

Then we have

$$\frac{d}{dt}\,\mathrm{Ad}_{\eta(t)^{-1}} = \mathrm{Ad}_{\eta(t)^{-1}}\,\mathcal{L}_{V(t)}, \tag{2.18}$$

and since $\mathcal{L}_V{}^\star$ is the L^2-metric adjoint of $\mathcal{L}_V$ in the space of divergence-free vector fields, we have by general properties of adjoints that

$$\frac{d}{dt}\,\mathrm{Ad}_{\eta(t)^{-1}}{}^\star = \mathcal{L}_{V(t)}{}^\star \mathrm{Ad}_{\eta(t)^{-1}}{}^\star, \tag{2.19}$$

where $\mathrm{Ad}_\eta{}^\star$ denotes the adjoint of the continuous operator Ad_η on $T_{\mathrm{id}}\mathcal{D}^0_\mu$, in the L^2 norm.

Explicitly, we can compute that this operator is

$$\mathrm{Ad}_\eta{}^\star(Z) = P\big((\eta^* Z^\flat)^\sharp\big) = P\big(D\eta^{\mathrm{T}}(Z \circ \eta)\big), \tag{2.20}$$

where η^* is the pullback operator on differential forms. (Notice that although Ad_η maps $T_{\mathrm{id}}\mathcal{D}^0_\mu$ to itself, the L^2 adjoint does not; thus we must compose with the projection P to get the $T_{\mathrm{id}}\mathcal{D}^0_\mu$ adjoint.)

For the Euler equation of ideal fluid mechanics, we know the vorticity 2-form $dV(t)^\flat$ is transported by the flow, i.e., that $dV(t)^\flat = [\eta(t)^{-1}]^* dV_o^\flat$ (see [E]). So $K_{V(t)}(Z) = P\big(\iota_Z[\eta(t)^{-1}]^* dV_o^\flat\big)$. Now for any vector field W, we have

$$\begin{aligned}\iota_Z(\eta^{-1})^* dV_o^\flat(W) = (\eta^{-1})^* dV_o^\flat(Z,W) &= dV_o^\flat(\eta_*^{-1}Z, \eta_*^{-1}W)\\ &= \iota_{\eta_*^{-1}Z} dV_o^\flat(\eta_*^{-1}W) = (\eta^{-1})^*\big(\iota_{\eta_*^{-1}Z} dV_o^\flat\big)(W).\end{aligned}$$

Thus by formula (2.20), we find that

$$K_{V(t)} = \mathrm{Ad}_{\eta(t)^{-1}}{}^\star \circ K_{V_o} \circ \mathrm{Ad}_{\eta(t)^{-1}}\,. \tag{2.21}$$

Computing the time-derivative of this operator, and using the equations (2.18) and (2.19), we obtain the formula (2.16), as desired. □

Using the splitting (2.15), the solution operator $F(t)$ of the Jacobi equation can then be written in the following convenient form. Unfortunately because the splitting loses one derivative (the equations are only defined on H^σ if V is in $H^{\sigma+1}$), we only get a result on H^σ rather than H^s. We will be able to compensate for this later.

Proposition 4. *If η is a geodesic curve in $\mathcal{D}^s_\mu(M)$, with $s > n/2+1$ and $s \ge \sigma+1$, then the map $F(t)$, which takes W_o to the Jacobi field $J(t)$ with initial conditions $J(0) = 0$ and $J'(0) = W_o$, extends to a continuous operator from $T_{id}\mathcal{D}^\sigma_\mu$ to $T_{\eta(t)}\mathcal{D}^\sigma_\mu$. In addition, we have the formula*

$$F(t) = D\eta(t)\big(\Omega(t) - \Gamma(t)\big). \tag{2.22}$$

Here $\Omega(t)\colon T_{id}\mathcal{D}^\sigma_\mu \to T_{id}\mathcal{D}^\sigma_\mu$ is a continuous operator, given by

$$\Omega(t) = \int_0^t \mathrm{Ad}_{\eta(\tau)^{-1}}\ \mathrm{Ad}_{\eta(\tau)^{-1}}{}^\star\, d\tau. \tag{2.23}$$

(The operators Ad_η and $\mathrm{Ad}_\eta{}^\star$ are as defined in Lemma 3.)

The operator $\Gamma(t)\colon T_{id}\mathcal{D}^\sigma_\mu \to T_{id}\mathcal{D}^\sigma_\mu$ is continuous and is given in terms of $F(t)$ by

$$\Gamma(t) = \int_0^t \mathrm{Ad}_{\eta(\tau)^{-1}}\ K_{V(\tau)}\, dR_{\eta^{-1}(\tau)}\, F(\tau)\, d\tau. \tag{2.24}$$

Proof. As in the proof of Lemma 3, we can rewrite the operators $\mathcal{L}_V$ and $\mathcal{L}_V{}^\star$ in terms of the push-forward Ad_η and its adjoint $\mathrm{Ad}_\eta{}^\star$. From equation (2.19), we can write

$$\frac{d}{dt}\mathrm{Ad}_{\eta(t)}{}^\star = -\mathrm{Ad}_{\eta(t)}{}^\star \mathcal{L}_{V(t)}{}^\star.$$

Using this equation and (2.18), the factored, right-translated Jacobi equation (2.15) can be written using operator derivatives as the pair of equations

$$\mathrm{Ad}_{\eta(t)^{-1}}{}^{\star}\,\frac{d}{dt}\big(\mathrm{Ad}_{\eta(t)}{}^{\star}W(t)\big) = 0, \tag{2.25}$$

$$\mathrm{Ad}_{\eta(t)}\,\frac{d}{dt}\big(\,\mathrm{Ad}_{\eta(t)^{-1}}\,Y(t)\big) + K_{V(t)}\big(Y(t)\big) = W(t). \tag{2.26}$$

The solution of (2.25) is obviously $W(t) = \mathrm{Ad}_{\eta(t)^{-1}}{}^{\star}W_o$, and from this we rewrite (2.26) as

$$\frac{d}{dt}\big(\,\mathrm{Ad}_{\eta(t)^{-1}}\,Y(t)\big) + \mathrm{Ad}_{\eta(t)^{-1}}\,K_{V(t)}\big(Y(t)\big) = \mathrm{Ad}_{\eta(t)^{-1}}\,\mathrm{Ad}_{\eta(t)^{-1}}{}^{\star}W_o. \tag{2.27}$$

This is a linear differential equation for $\mathrm{Ad}_{\eta(t)^{-1}}\,Y(t)$ on $T_{\mathrm{id}}\mathcal{D}^0_\mu$. Since η, η^{-1}, and V are all H^s, we know $\mathrm{Ad}_{\eta^{-1}}$, $\mathrm{Ad}_{\eta^{-1}}{}^{\star}$, and K_V are all continuous operators on $T_{\mathrm{id}}\mathcal{D}^\sigma_\mu$. Therefore there is a unique solution $Y(t)$ in $T_{\mathrm{id}}\mathcal{D}^\sigma_\mu$ with $Y(0) = 0$, defined for as long as $\eta(t)$ and $V(t)$ are defined. Since $F(t)(W_o) = Y(t)\circ\eta(t)$, this shows that $F(t)$ is defined on all of $T_{\mathrm{id}}\mathcal{D}^\sigma_\mu$.

Now, instead of actually solving (2.27), we simply integrate both sides in time and obtain

$$\mathrm{Ad}_{\eta(t)^{-1}}\,Y(t) = \Omega(t)(W_o) - \Gamma(t)(W_o).$$

Using $J(t) = Y(t)\circ\eta(t)$ and the formula (2.17), we get the formula (2.22) as desired.

Continuity of the operators $\Omega(t)$ and $\Gamma(t)$ is clear from their definitions, since η^{-1} and V are both in $H^{\sigma+1}$. □

The basic idea of the Fredholmness proof is to use the decomposition (2.22), showing that $\Omega(t)$ is invertible and that $\Gamma(t)$ is compact. We will do this in the next section.

3. Proof of Fredholmness

We first establish that $\Omega(t)$ is invertible on $T_{\mathrm{id}}\mathcal{D}^\sigma_\mu(M)$. If $\sigma = 0$, this is true for M of any dimension and possibly with boundary. On the other hand, if $\sigma > 0$, then it is true for a manifold of any dimension, without boundary. First the $\sigma = 0$ result.

Proposition 5. *Suppose M is a compact manifold of any dimension, possibly with boundary. If $V_o \in T_{id}\mathcal{D}^s_\mu$ with $s > n/2+1$ and $\eta(t) = \exp_{id}(tV_o)$, then the operator $\Omega(t)$ defined by equation* (2.23) *is positive-definite on $T_{id}\mathcal{D}^0_\mu$, satisfying the estimate*

$$\langle\langle W_o, \Omega(t)(W_o)\rangle\rangle_{L^2} \geq C(t)\langle\langle W_o, W_o\rangle\rangle_{L^2}, \tag{3.28}$$

with $C(t) = \displaystyle\int_0^t \frac{d\tau}{\|D\eta(\tau)^T D\eta(\tau)\|_{L^\infty}}$. Consequently, $\Omega(t)$ is also invertible on $T_{id}\mathcal{D}^0_\mu$.

Proof. We have

$$\langle\langle W_o, \Omega(t)(W_o)\rangle\rangle_{L^2} = \int_0^t \langle\langle W_o, \mathrm{Ad}_{\eta(\tau)^{-1}}\mathrm{Ad}_{\eta(\tau)^{-1}}{}^{\star}(W_o)\rangle\rangle_{L^2}\,d\tau$$
$$= \int_0^t \|\mathrm{Ad}_{\eta(\tau)^{-1}}{}^{\star}(W_o)\|_{L^2}^2\,d\tau \geq \left(\int_0^t \frac{1}{\|\mathrm{Ad}_{\eta(\tau)}{}^{\star}\|_{L^2}^2}\,d\tau\right)\|W_o\|_{L^2}^2.$$

We can compute, using formula (2.17) for the push-forward map, that

$$\|\mathrm{Ad}_{\eta(\tau)}{}^{\star}\|_{L^2}^2 = \|\mathrm{Ad}_{\eta(\tau)}\|_{L^2}^2 \leq \|D\eta(\tau)^T D\eta(\tau)\|_{L^\infty},$$

the L^∞ norm denoting the maximum on M of the largest eigenvalue of the symmetric matrix $D\eta^T D\eta$, which is well-defined since η is C^1. The inequality (3.28) then follows.

Now since $\Omega(t)$ is positive-definite, it must have empty kernel. Since $\Omega(t)$ is self-adjoint, it must also have empty cokernel. And by the Schwartz inequality, we have

$$C(t)\|W_o\|_{L^2} \leq \|\Omega(t)(W_o)\|_{L^2},$$

which implies that $\Omega(t)$ has closed range, and is hence surjective. So $\Omega(t)$ is invertible on $T_{\mathrm{id}}\mathcal{D}_\mu^0$. □

To define the Sobolev topology on $T_\eta\mathcal{D}_\mu^\sigma(M)$, where M has no boundary, we set up coordinate systems on a partition of unity $\{O_j\}$ of M, and then for any element $X\circ\eta \in T_\eta\mathcal{D}_\mu^\sigma$, write $X\circ\eta = \sum_k X^k(\eta)\,\partial_k$ and define the Sobolev norm as

$$\|X\circ\eta\|_{H^\sigma} = \sum_j\sum_k\sum_{|\alpha|\leq\sigma}\|\partial_{x^1}^{\alpha_1}\partial_{x^2}^{\alpha_2}\cdots\partial_{x^n}^{\alpha_n}X^k(\eta)\|_{L^2(O_j)}, \tag{3.29}$$

where $|\sigma| = \alpha_1+\alpha_2+\cdots+\alpha_n$. To simplify notation, we will abbreviate this as

$$\|X\circ\eta\|_{H^\sigma} = \sum\|\partial^\alpha X(\eta)\|_{L^2}.$$

See Ebin-Marsden [EMa] for details of these constructions.

Lemma 6. *Suppose M is a compact manifold of dimension n, without boundary, and O is a coordinate patch in M. Suppose η is a C^∞ volume-preserving diffeomorphism. Then for any multi-index α with $|\alpha|\leq\sigma$ and any W in H^σ, we have the estimate*

$$\|[\partial^\alpha, P_\eta](W)\|_{L^2(O)} \leq B_\alpha\|W\|_{H^{\sigma-1}(O)}$$

for some constant B_α.

Proof. First consider $[\partial_{x^i}, P_\eta]$. If X is an H^σ vector field (not necessarily divergence-free), then we can write $X = U + \nabla f$, where $U \in T_{\mathrm{id}}\mathcal{D}_\mu^\sigma$ and f is an $H^{\sigma+1}$ function. Then

$$[\partial_{x^i}, P_\eta](X\circ\eta) = \partial_{x^i}(U\circ\eta) - P\big(\partial_{x^i}(X\circ\eta)\circ\eta^{-1}\big)\circ\eta.$$

We will prove first that

$$\|[\partial_{x^i}, P_\eta](X\circ\eta)\|_{L^2} \leq b_i\|X\|_{L^2}. \tag{3.30}$$

Write $N_i^j = \frac{\partial \eta^j}{\partial x^i} \circ \eta^{-1}$. Then, with g^{ij} denoting the components of the inverse metric, we have

$$\begin{aligned}\partial_{x^i}(X^k \circ \eta)\circ\eta^{-1}\,\partial_k &= \left(N_i^j \frac{\partial X^k}{\partial x^j}\right)\partial_k \\ &= \left(N_i^j\frac{\partial U^k}{\partial x^j} + N_i^j \frac{\partial g^{kl}}{\partial x^j}\frac{\partial f}{\partial x^l} + N_i^j g^{kl}\frac{\partial^2 f}{\partial x^j \partial x^l}\right)\partial_k \\ &= \left(N_i^j\frac{\partial U^k}{\partial x^j} + N_i^j \frac{\partial g^{kl}}{\partial x^j}\frac{\partial f}{\partial x^l} - g^{kl}\frac{\partial f}{\partial x^j}\frac{\partial N_i^j}{\partial x^l}\right)\partial_k + \nabla\left(N_i^j \frac{\partial f}{\partial x^j}\right).\end{aligned}$$

Since the projection of a gradient is zero, we have

$$[\partial_{x^i}, P_\eta](X\circ\eta) = Q\left(N_i^j \frac{\partial U^k}{\partial x^j}\,\partial_k\right)\circ\eta - P\left(\left(N_i^j \frac{\partial g^{kl}}{\partial x^j}\frac{\partial f}{\partial x^l} - g^{kl}\frac{\partial f}{\partial x^j}\frac{\partial N_i^j}{\partial x^l}\right)\partial_k\right)\circ\eta.$$

The L^2 norm of the second term is bounded by the L^2 norm of first derivatives of f, which are in turn bounded by the L^2 norm of X. It remains to bound the L^2 norm of the first term by the L^2 norm of X.

We recall that for any vector field Z, we have

$$Q(Z) = \nabla\Delta^{-1}\operatorname{div} Z,$$

where the inverse Laplacian is uniquely determined up to a constant since the manifold has no boundary. So we simply compute $\operatorname{div}\left(N_i^j \frac{\partial U^k}{\partial x^j}\,\partial_k\right)$. If in coordinates the volume form is $\mu = \varphi\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$, we have

$$\operatorname{div}\left(N_i^j \frac{\partial U^k}{\partial x^j}\,\partial_k\right) = \frac{1}{\varphi}\frac{\partial}{\partial x^k}\left(\varphi N_i^j \frac{\partial U^k}{\partial x^j}\right) = \frac{\partial N_i^j}{\partial x^k}\frac{\partial U^k}{\partial x^j} - \frac{1}{\varphi} N_i^j \frac{\partial}{\partial x^k}\left(U^k \frac{\partial \varphi}{\partial x^j}\right),$$

using the fact that U is divergence-free. Since this expression only involves first derivatives of U, we have

$$\|Q(Z)\|_{L^2} \le \|\Delta^{-1}\operatorname{div} Z\|_{H^1} \le C\|U\|_{L^2} \le C\|X\|_{L^2}.$$

So we therefore can conclude the formula (3.30), as desired.

Inductively, the formula (3.30) implies that for any multi-index α with no more than σ terms, we will have

$$\|[\partial^\alpha, P_\eta](W)\|_{L^2} \le B_\alpha \|W\|_{H^{\sigma-1}},$$

as desired. □

Now we prove invertibility of $\Omega(t)$ on $T_{\mathrm{id}}\mathcal{D}^\sigma_\mu$. For convenience we assume that $\eta(t)$ is C^∞, but this will not affect the proof of Theorem 1.

Proposition 7. *Suppose M is a compact manifold of dimension n without boundary. If V_o is a smooth, divergence-free vector field and $\eta(t) = \exp_{id}(tV_o)$, then $\Omega(t)$ defined by equation* (2.23) *satisfies the estimate*

$$\|\Omega(t)(W_o)\|_{H^\sigma} \ge C(t)\|W_o\|_{H^\sigma} - K\|W_o\|_{H^{\sigma-1}}, \tag{3.31}$$

where $C(t) = \displaystyle\int_0^t \frac{d\tau}{\|D\eta(\tau)^T D\eta(\tau)\|_{L^\infty}}$ *and K is some constant. Consequently $\Omega(t)$ has closed range on $T_{id}\mathcal{D}^\sigma_\mu$ and hence is invertible.*

Proof. To obtain the H^σ estimate, we use the L^2 bound on Ω from formula (3.28) to get

$$\begin{aligned}\|\Omega(t)(W_o)\|_{H^\sigma} &= \sum \|\partial^\alpha \Omega(t)(W_o)\|_{L^2} \\ &\geq \sum \|\Omega(t)(\partial^\alpha W_o)\|_{L^2} - \sum \|[\partial^\alpha, \Omega(t)](W_o)\|_{L^2} \\ &\geq C(t)\|W_o\|_{H^\sigma} - \sum \|[\partial^\alpha, \Omega(t)](W_o)\|_{L^2}.\end{aligned} \tag{3.32}$$

Now we want to show that the commutator term is bounded in $H^{\sigma-1}$. Using formulas (2.18) and (2.19) in the definition (2.23) of $\Omega(t)$, we can write

$$\Omega(t) = \int_0^t D\eta^{-1}(\tau)\, dR_{\eta(\tau)}\; P\; dR_{\eta^{-1}(\tau)}\, D\eta^{-1}(\tau)^{\mathrm{T}}\, d\tau. \tag{3.33}$$

As a consequence, it is obviously enough to show that for each fixed τ, the commutator

$$\sum \|[\partial^\alpha, D\eta^{-1}(\tau)\; P_{\eta(\tau)}\; D\eta^{-1}(\tau)^{\mathrm{T}}](W_o)\|_{L^2}$$

is bounded in terms of the $H^{\sigma-1}$ norm of W_o, where we used the formula $P_\eta = dR_\eta \circ P \circ dR_{\eta^{-1}}$ to simplify notation. Now since

$$\begin{aligned}\sum \|[\partial^\alpha, D\eta^{-1}\; P_\eta\; (D\eta^{-1})^{\mathrm{T}}](W_o)\|_{L^2} &\leq \sum \|[\partial^\alpha, D\eta^{-1}]\; P_\eta\; (D\eta^{-1})^{\mathrm{T}}(W_o)\|_{L^2} \\ &+ \sum \|D\eta^{-1}\; [\partial^\alpha, P_\eta]\; (D\eta^{-1})^{\mathrm{T}}(W_o)\|_{L^2} + \sum \|D\eta^{-1}\; P_\eta\; [\partial^\alpha, (D\eta^{-1})^{\mathrm{T}}](W_o)\|_{L^2},\end{aligned} \tag{3.34}$$

it is enough to prove that the L^2 norms $\|[\partial^\alpha, D\eta^{-1}](W)]\|_{L^2}$, $\|[\partial^\alpha, P_\eta](W)\|_{L^2}$, and $\|[\partial^\alpha, (D\eta^{-1})^{\mathrm{T}}](W)\|_{L^2}$ can all be bounded by $\|W\|_{H^{\sigma-1}}$.

For the first and third terms, we notice that the operators $D\eta^{-1}$ and $(D\eta^{-1})^{\mathrm{T}}$ are simply matrices of smooth functions. The commutator of an σ-order derivative operator and a multiplication operator is a $(\sigma-1)$-order derivative operator, by the Leibniz rule, and so we are done with these. For the second term, we use Lemma 6. We have thus established the estimate (3.31), and thus we know that $\Omega(t)$ has closed range.

To establish invertibility of $\Omega(t)$ on $T_{\mathrm{id}}\mathcal{D}^\sigma_\mu$, we first use the fact from Proposition 5 that $\Omega(t)$ is invertible as a map from $T_{\mathrm{id}}\mathcal{D}^0_\mu$. So if $Y \in T_{\mathrm{id}}\mathcal{D}^\sigma_\mu$, we choose $X \in T_{\mathrm{id}}\mathcal{D}^0_\mu$ such that $\Omega(X) = Y$, and prove inductively using the estimate (3.31) that the H^k norms of X are bounded for $k \leq \sigma$. Thus $X \in T_{\mathrm{id}}\mathcal{D}^\sigma_\mu$. □

Remark. The reason Proposition 7 fails if M has a boundary is that Lemma 6 fails. If in coordinates the boundary is $y = 0$, then $\|[\partial_y, P]W\|_{L^2} = \|Q\partial_y P\|_{L^2}$ is not bounded by $\|W\|_{L^2}$. The problem is that although $\partial_y W$ is divergence-free if W is, it need not be tangent to the boundary.

On the other hand, if we used a weaker Sobolev topology involving only the derivatives tangent to the boundary, Lemma 6 would be valid, as would Proposition 7.

All of the preceding is true for n-dimensional M, but the proof of Fredholmness works only in two dimensions, and for the rest of this section we will specialize to that situation.

The essential difference between the two-dimensional and the three-dimensional cases is the following compactness result.

Proposition 8. *Suppose M is a two-dimensional manifold, possibly with boundary. If $s > 2$ and $s \geq \sigma + 1$, then for any H^s vector field V, divergence-free and tangent to ∂M, the operator*

$$K_V \colon T_{id}\mathcal{D}^\sigma_\mu(M) \to T_{id}\mathcal{D}^\sigma_\mu(M),$$

defined by the formula (2.13)*, is compact.*

Proof. We can approximate V in the H^s norm by a sequence of smooth vector fields V_k, such that $K_{V_k} \to K_V$ in the H^σ operator norm. Since a limit of compact operators is also compact, it is enough to prove that K_V is compact if V is smooth.

In two dimensions, the operator K_V can be simplified to

$$K_V(Z) = P\big((\operatorname{curl} V) \star Z\big),$$

where $\operatorname{curl} V$ is a function (the vorticity function of the vector field V) and $\star Z$ denotes the vector field obtained by rotating Z by 90° in each tangent space.

We note that any $Z \in T_{\mathrm{id}}\mathcal{D}^\sigma_\mu$ may be written as

$$Z = \pi_{\mathrm{s}}(Z) + \pi_{\mathrm{h}}(Z) = \star\nabla f + \alpha, \tag{3.35}$$

where $f \colon M \to \mathbb{R}$ is an $H^{\sigma+1}$ function which vanishes on the boundary, and α is a smooth harmonic vector field, i.e., a field satisfying $\operatorname{div}\alpha = \operatorname{curl}\alpha = 0$ and tangent to the boundary. The projections π_{s} and π_{h} are both continuous in the H^σ topology.

The space of harmonic vector fields tangent to the boundary has finite dimension, and an immediate consequence is that π_{h} is compact. Thus $K_V \circ \pi_{\mathrm{h}}$ is also compact, and since

$$K_V = K_V \circ \pi_{\mathrm{s}} + K_V \circ \pi_{\mathrm{h}},$$

it is now sufficient to prove $K_V \circ \pi_{\mathrm{s}}$ is compact.

We compute

$$\begin{aligned} K_V \circ \pi_{\mathrm{s}}(Z) = K_V(\star\nabla f) &= -P\big((\operatorname{curl} V)\nabla f\big) \\ &= P\big(-\nabla(f \operatorname{curl} V) + f\,\nabla \operatorname{curl} V\big) = P\big(f\,\nabla\operatorname{curl} V\big), \end{aligned} \tag{3.36}$$

since the projection of a gradient is zero. Since $\nabla\operatorname{curl} V$ is a smooth vector field, we know $f\,\nabla\operatorname{curl} V$ is an $H^{\sigma+1}$ vector field. Thus the map $Z \mapsto f\nabla\operatorname{curl} V$, being a continuous map from H^σ vector fields to $H^{\sigma+1}$ vector fields, is compact by Rellich's Lemma. Since P is continuous, we see $K_V \circ \pi_{\mathrm{s}}$ is a composition of a continuous and a compact operator, and hence compact. □

Corollary 9. *Suppose M is a two-dimensional manifold, possibly with boundary. Suppose $s > 2$ and $s \geq \sigma + 1$. Let $V_o \in T_{id}\mathcal{D}^s_\mu$, and let $\eta(t) = \exp_{id}(tV_o)$. Then the operator $\Gamma(t)$ defined by equation* (2.24) *is compact on $T_{id}\mathcal{D}^\sigma_\mu$.*

Proof. Since we know the operators $F(\tau)$ and $\eta^{-1}(\tau)_*$ are both continuous and $K_{V(\tau)}$ is compact for each τ, the composition appearing in the integral (2.24) which defines $\Gamma(t)$ is compact for each τ. Then the integral $\Gamma(t)$, as a limit of sums of compact operators, is also compact. □

Proposition 5 and Corollary 9 now allow us to prove Theorem 2.

Proof of Theorem 2. Since $\Omega(t)$ is invertible and $\Gamma(t)$ is compact on $T_{\mathrm{id}}\mathcal{D}^0_\mu$, we know $\Omega(t) - \Gamma(t)$ is Fredholm on $T_{\mathrm{id}}\mathcal{D}^0_\mu$. Since $dL_{\eta(t)} = D\eta(t)$ is continuous and invertible on $T_{\mathrm{id}}\mathcal{D}^0_\mu$, the decomposition (2.22) proves that $F(t)$ is Fredholm, and thus that $d\exp_{\mathrm{id}}(tV_o)$ is as well.

The index is a continuous function on the space of Fredholm operators. When $t = 0$, $d\exp_{\mathrm{id}}(0)$ is the identity map, which has index zero. Thus the index is zero for all t. □

The proof of Fredholmness in H^s is not quite as simple, since if η is H^s, the decomposition (2.22) only works in H^{s-1}. Nonetheless we can approximate η by a smoother $\tilde{\eta}$ to obtain the result.

Proof of Theorem 1. It is enough to prove that $F(t) = d\exp_{\mathrm{id}}(tV_o)$ has closed range and finite-dimensional kernel. If these conditions hold, then $F(t)$ is semi-Fredholm and we can compute its index. Since the index is continuous on the space of semi-Fredholm maps, we must have the same index for all time. Since $F(0)$ is the identity, with index zero, we will be able to conclude that $F(t)$ is Fredholm of index zero for all t.

Finite-dimensionality of the kernel in $T_{\mathrm{id}}\mathcal{D}^s_\mu$ follows from finite-dimensionality of the kernel in $T_{\mathrm{id}}\mathcal{D}^0_\mu$, since the former kernel is a subset of the latter. So the proof will be complete once we prove that $F(t)$ has closed range in $T_{\mathrm{id}}\mathcal{D}^s_\mu$.

It will be sufficient to establish an estimate of the form

$$A\|W_o\|_{H^s} \le \|F(t)(W_o)\|_{H^s} + B\|W_o\|_{H^{s-1}} + \|\Phi(W_o)\|_{H^s}, \tag{3.37}$$

for some positive constants A and B, where Φ is a compact operator on H^s.

First we choose a C^∞ vector field $\tilde{V}_o$, close to V_o in the H^s norm, in a sense to be specified later. For such a vector field, the geodesic $\tilde{\eta}(t)$ is also smooth and is defined for all time. Then from the decomposition (2.22) we can write

$$\tilde{F}(t) = D\tilde{\eta}\big(\tilde{\Omega}(t) - \tilde{\Gamma}(t)\big),$$

and this formula is valid on $T_{\mathrm{id}}\mathcal{D}^s_\mu(M)$ as well as on $T_{\mathrm{id}}\mathcal{D}^0_\mu(M)$.

We observe that since the geodesic and Jacobi equations are smooth on $\mathcal{D}^s_\mu$, the solutions depend continuously on the initial conditions. Thus $\tilde{F}(t)$ is close to $F(t)$ in the H^s operator norm and $\tilde{\eta}(t)$ is close to $\eta(t)$ in H^s. We then have

$$\begin{aligned} \|F(t)(W_o)\|_{H^s} &\ge \|\tilde{F}(t)(W_o)\|_{H^s} - \|\tilde{F}(t) - F(t)\|_{H^s}\|W_o\|_{H^s} \\ &\ge \|D\tilde{\eta}\,\tilde{\Omega}(W_o)\|_{H^s} - \|D\tilde{\eta}\,\tilde{\Gamma}(W_o)\|_{H^s} - \|\tilde{F}(t) - F(t)\|_{H^s}\|W_o\|_{H^s}. \end{aligned} \tag{3.38}$$

(The subtraction of $F(t)$ from $\tilde{F}(t)$ is not well-defined without the chosen coordinate system, since the two maps generally have different tangent spaces as ranges. Here we simply mean the subtraction of the components in the coordinate system, which is well-defined.)

From Proposition 7, we have the estimate

$$\|\tilde{\Omega}(t)(W_o)\|_{H^s} \geq \tilde{C}(t)\|W_o\|_{H^s} - K\|W_o\|_{H^{s-1}}, \tag{3.39}$$

where

$$\tilde{C}(t) = \int_0^t \frac{d\tau}{\|D\tilde{\eta}(\tau)^T D\tilde{\eta}(\tau)\|_{L^\infty}}.$$

Inserting (3.39) into (3.38) we obtain, with $\Phi = D\tilde{\eta}(t)\tilde{\Gamma}(t)$ and some constant B, the inequality

$$\begin{aligned} \|F(t)(W_o)\|_{H^s} &\geq \frac{\tilde{C}}{\|D\tilde{\eta}^{-1}\|_{L^\infty}}\|W_o\|_{H^s} - \|\tilde{F}(t) - F(t)\|_{H^s}\|W_o\|_{H^s} \\ &\qquad - B\|W_o\|_{H^{s-1}} - \|\Phi(W_o)\|_{H^s} \\ &\geq A\|W_o\|_{H^s} - B\|W_o\|_{H^{s-1}} - \|\Phi(W_o)\|_{H^s}, \end{aligned} \tag{3.40}$$

where

$$A = \frac{C}{\|D\eta^{-1}\|_{L^\infty}} - \Big(\frac{C}{\|D\eta^{-1}\|_{L^\infty}} - \frac{\tilde{C}}{\|D\tilde{\eta}^{-1}\|_{L^\infty}}\Big) - \|\tilde{F}(t) - F(t)\|_{H^s}. \tag{3.41}$$

The last two terms in (3.40) are H^s norms of compact operators on $T_{\mathrm{id}}\mathcal{D}^s_\mu(M)$, and so we will have obtained the desired estimate (3.37) if we choose $\tilde{V}_o$ close enough to V_o that the number A is positive.

The number C given by (3.28) depends only on the C^1 norm of η, as does the number $\|D\eta^{-1}\|_{L^\infty}$. So if $\tilde{V}_o$ is close enough to V_o in H^s, then $\tilde{C}$ will be close to C and $\|D\tilde{\eta}^{-1}\|_{L^\infty}$ will be close to $\|D\eta^{-1}\|_{L^\infty}$. In addition $\|\tilde{F}(t) - F(t)\|_{H^s}$ will be close to zero. Thus A can be made positive, so that the estimate (3.37) will be satisfied. So $F(t) = d\exp_{\mathrm{id}}(tV_o)$ must have closed range in the H^s topology.

The proof of Theorem 1 is now complete. □

4. A counterexample in three dimensions

In three dimensions, the operator K_V takes the form

$$K_V(Z) = P(\iota_Z dV^\flat)^\sharp = -P(Z \times \operatorname{curl} V),$$

which is typically not a compact operator. Thus the proof of Fredholmness fails even in $T_{\mathrm{id}}\mathcal{D}^0_\mu(M)$. In fact, the result is false, as we will show with a very explicit example.

Let M be the solid cylinder in $\mathbb{R}^3$ of radius 1 and height 2π, and identify the top and bottom surfaces to obtain a solid torus. Use cylindrical coordinates $\{r, \theta, z\}$. Let $V = \frac{\partial}{\partial\theta}$ be a rigid rotation of the cylinder. We can easily verify that V is a steady solution of the Euler equation (1.2), with pressure function $p(r) = \frac{1}{2}r^2$.

Theorem 10. *For M the solid cylinder as above and $V = \partial_\theta$, the differential of the exponential map is not Fredholm at the point $\pi V \in T_{id}\mathcal{D}^s_\mu(M)$.*

Proof. Our method will be to compute all of the conjugate points explicitly. We will demonstrate that, along the geodesic $\exp_{\mathrm{id}}(tV)$, a sequence of conjugate point locations decreases to π. Then we show that as a result, the differential of the exponential map is not closed at πV.

We will solve the right-translated Jacobi equation in the form (2.14). We observe that $\mathcal{L}_V = \partial_\theta$ and $\mathcal{L}_V{}^\star = -\partial_\theta$, where ∂_θ is the componentwise partial derivative. We also compute that since $\operatorname{curl}\partial_\theta = 2\,\partial_z$,

$$\operatorname{curl} K_V(Z) = \operatorname{curl}(2\,\partial_z \times Z) = -[2\,\partial_z, Z] + 2(\operatorname{div} Z)\,\partial_z - 2(\operatorname{div}\partial_z)Z = -2\,\partial_z Z,$$

and thus $K_V = -2\,\partial_z \operatorname{curl}^{-1}$, where curl^{-1} is the inverse operator defined on the space of divergence-free vector fields. (By convention we will take $\operatorname{curl}^{-1}(\partial_z) = 0$.) Equation (2.14) then becomes

$$(\partial_t + \partial_\theta - 2\,\partial_z \operatorname{curl}^{-1})(\partial_t + \partial_\theta)Y(t) = 0. \tag{4.42}$$

Recall that there is a basis for the divergence-free vector fields tangent to the boundary of the cylinder, given by $U_{000} = \partial_z$ and

$$U_{kmn} = \lambda_{kmn}\nabla\Phi_{kmn} \times \partial_z + \operatorname{curl}(\nabla\Phi_{lmn} \times \partial_z), \qquad k \in \mathbb{Z}\backslash\{0\}, m \in \mathbb{Z}, n \in \mathbb{Z}$$

where $\lambda_{kmn} = \operatorname{sgn}(k)\sqrt{\alpha_{kmn}^2 + m^2}$ and

$$\Phi_{kmn} = J_n(\alpha_{kmn} r)\, e^{in\theta}\, e^{imz},$$

with J_n the Bessel function of order n. The number α_{kmn} is determined by the boundary condition

$$\lambda_{kmn} n J_n(\alpha_{kmn}) - m J_n'(\alpha_{kmn}) = 0, \qquad k \neq 0, \quad m \text{ and } n \text{ not both zero,} \tag{4.43}$$

or, if $m = n = 0$, the condition $J_1(\alpha_{k00}) = 0$ for $k \geq 1$. We set $\alpha_{000} = 0$ so that $\lambda_{000} = 0$ (the field U_{000} being the only harmonic vector field on M). We have

$$\operatorname{curl} U_{kmn} = \lambda_{kmn} U_{kmn}.$$

(This is the Chandrasekhar-Kendall [CK] construction of the eigenfields of curl on the cylinder; see Yoshida [Y] for a proof of completeness.)

Thus if we expand $Y(t) = \sum y_{kmn}(t) U_{kmn}$, then by equation (4.42), the coefficients y_{kmn} satisfy the ordinary differential equation

$$\left(\frac{d}{dt} + in - \frac{2im}{\lambda_{kmn}}\right)\left(\frac{d}{dt} + in\right) y_{kmn}(t) = 0, \quad (k \neq 0),$$

and the solutions with $y_{kmn}(0) = 0$ and $y'_{kmn}(0) = w_{kmn}(0)$ are given by

$$y_{kmn}(t) = w_{kmn}(0)\frac{\lambda_{kmn}}{m}\sin\left(\frac{mt}{\lambda_{kmn}}\right)e^{-int}e^{imt/\lambda_{kmn}}, \qquad k \neq 0, m \neq 0.$$

If $m = 0$ and $k \neq 0$, then the solutions are $y_{k0n}(t) = w_{k0n}(0)\, te^{-int}$. If $k = 0$, then the solution is $y_{000}(t) = w_{000}(0)\, t$. (Of course, we will actually be interested in the real parts $\Re(y_{kmn}(t)U_{kmn})$, but for the purpose of studying the zeroes, we only need to be concerned with the amplitude of this complex expression.)

We see that the monoconjugate points occur at times $t \in C$, where

$$C = \left\{ j\pi \left| \frac{\lambda_{kmn}}{m} \right| \Big| j \in \mathbb{N}, k, m \in \mathbb{Z}\backslash\{0\}, n \in \mathbb{Z} \right\}.$$

All points contained in C are both monoconjugate and epiconjugate points. In the simplest case $n = 0$, $j = 1$, $k, m \in \mathbb{N}$, we have the monoconjugate points

$$\tau_{km0} = \frac{\lambda_{km0}}{m} = \frac{\pi\sqrt{\alpha_{k00}^2 + m^2}}{m},$$

since the boundary condition (4.43), with $n = 0$, ends up not depending on m. Thus as $m \to \infty$, we see $\tau_{km0} \to \pi$, so that C is not discrete. Observe also that $\pi \notin C$, since for any $k \neq 0$, $m \neq 0$, $n \in \mathbb{Z}$, the curl eigenvalue satisfies $|\lambda_{kmn}| = \sqrt{\alpha_{kmn}^2 + m^2} > m$.

We will show that the range of the differential of the exponential map is not closed at πV; in other words, we will construct a sequence of Jacobi fields $Y^{(M)}$ such that $Y^{(M)}(\pi)$ converges but the initial conditions $W_o^{(M)} = \nabla_{\eta'(0)} Y^{(M)}$ do not converge.

First notice that the eigenfields U_{kmn} of curl are orthogonal in L^2, since curl is self-adjoint. We can generate the Sobolev topology by setting the H^s norm of any $W = \sum_{kmn} w_{kmn} U_{kmn} \in T_{\mathrm{id}}\mathcal{D}_\mu^s(M)$ to be

$$\langle\langle W, W \rangle\rangle_{H^s} = \sum_{k,m,n\in\mathbb{Z}} (1 + \lambda_{kmn}^2)^s \|U_{kmn}\|_{L^2}^2 |w_{kmn}|^2.$$

Fix an s, and consider the sequence

$$W_o{}^{(M)} = \sum_{m=1}^{M} \frac{1}{\|U_{1m0}\|_{L^2}} \frac{1}{(1 + \lambda_{1m0}^2)^{s/2}} U_{1m0}.$$

This sequence does not converge in H^s as $M \to \infty$, since we have

$$\|W_o^{(M)}\|_{H^s}^2 = M.$$

On the other hand, the corresponding Jacobi field solution $Y^{(M)}(\pi)$ has squared H^s norm given by

$$\|Y^{(M)}(\pi)\|_{H^s}^2 = \sum_{m=1}^{M} \frac{\lambda_{1m0}^2}{m^2} \sin^2\Big(\frac{\pi m}{\lambda_{m10}}\Big)$$

We estimate the size of the m^{th} term, which looks, as $m \to \infty$, like

$$\frac{\alpha_{100}^2 + m^2}{m^2} \sin^2\Big(\frac{\pi m}{\sqrt{\alpha_{100}^2 + m^2}}\Big) \approx \frac{\pi^2 \alpha_{100}^4}{4m^4}.$$

Thus the series converges by comparison to $\sum_m m^{-4}$. So as $M \to \infty$, the norm $\|Y^{(M)}(\pi)\|_{H^s}$ remains bounded and thus $Y^{(M)}(\pi)$ converges in H^s.

Thus the map $W_o \mapsto Y(\pi)$ is injective but not surjective, and thus cannot have closed range. Thus nor does $d\exp_{\mathrm{id}}(\pi V)$, so it is not Fredholm. □

The same technique can be used to prove that if t is any cluster point of the set C, the differential of the exponential map will not be Fredholm at tV. It is not clear whether there are any other cluster points besides the integer multiples of π. Observe that in the proof we actually demonstrated that $\exp_{\mathrm{id}}(\pi V)$ was an epiconjugate point but not a monoconjugate point. It is interesting that in this case $\exp_{\mathrm{id}}(\pi V)$ is also a cut point, since $\exp_{\mathrm{id}}(-\pi V) = \exp_{\mathrm{id}}(\pi V)$ (i.e., the rotation geodesic minimizes whether one rotates clockwise or counterclockwise).

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