ONE-PARAMETER SOLUTIONS OF THE EULER-ARNOLD EQUATION ON THE CONTACTOMORPHISM GROUP

Stephen C. Preston

Department of Mathematics
University of Colorado
Boulder, CO 80309-0395, USA

Alejandro Sarria

Department of Mathematics
University of Colorado
Boulder, CO 80309-0395, USA

(Communicated by the associate editor name)

Abstract. We study solutions of the equation

$$g_t - g_{tyy} + 4g^2 - 4gg_{yy} = ygg_{yyy} - yg_yg_{yy}, \qquad y \in \mathbb{R},$$

which arises by considering solutions of the Euler-Arnold equation on a contactomorphism group when the stream function is of the form $f(t, x, y, z) = zg(t, y)$. The equation is analogous to both the Camassa-Holm equation and the Proudman-Johnson equation. We write the equation as an ODE in a Banach space to establish local existence, and we describe conditions leading to global existence and conditions leading to blowup in finite time.

1. **Introduction.** In this brief note we study regularity of solutions to the Cauchy problem

$$\begin{cases} g_t - g_{yyt} + 4g^2 - 4gg_{yy} = ygg_{yyy} - yg_yg_{yy}, & (t, y) \in \mathbb{R}^+ \times \mathbb{R}. \\ g(0, y) = g_0(y), & y \in \mathbb{R}. \end{cases} \tag{1}$$

We prove the following result.

Main Theorem. *Let $\phi_0(y) = g_0(y) - g_0''(y)$. Suppose g_0 is C^2 and ϕ_0 satisfies the decay condition $\phi_0(y) = O(1/y^2)$ as $|y| \to \infty$. Then there is a $T > 0$ such that there exists a unique solution of (1) on $[0, T) \times \mathbb{R}$ with $g(t) \in C^2$ for each t. If ϕ_0 (and hence g_0) is nonnegative, then solutions exist globally. If g_0 is even and negative, then solutions blow up at some T in the sense that $g(t, y) \to -\infty$ as $t \nearrow T$ for every $y \in \mathbb{R}$.*

Equation (1) is a special case of the Euler-Arnold equation on the contactomorphism group $\mathrm{Diff}_\theta(M)$:

$$m_t + u(m) + (n + 2)\lambda m = 0, \tag{2}$$

where M is a Riemannian manifold of odd dimension $2n + 1$ with a 1-form θ satisfying $\theta \wedge (d\theta)^n \neq 0$. Here $f \colon M \to \mathbb{R}$ is a stream function, while $u = S_\theta f$ is a contact vector field (satisfying the condition that $\mathcal{L}_u\theta$ is proportional to θ); the field u is uniquely determined by f via the condition $f = \theta(u)$. We denote by λ the function such that $\mathcal{L}_u\theta = \lambda\theta$, and we

write $\widetilde{S_\theta} f = (S_\theta f, \lambda)$. A Riemannian metric on M determines a right-invariant Riemannian metric on $\text{Diff}(M) \times C^\infty(M)$, which allows us to define $m = \widetilde{S_\theta}^* \widetilde{S_\theta} f$, which is called the contact Laplacian. See [10] for the derivation and local well-posedness theory of this equation when M is compact, along with other examples.

When $n = 0$ (so that M is $\mathbb{R}$ or S^1) and $\theta = dx$, we have $u = f \partial_x$, $\lambda = f_x$, and $m = f - f_{xx}$, and equation (2) becomes the Camassa-Holm (CH) equation [4]

$$f_t - f_{txx} + 3ff_x - 2f_x f_{xx} - ff_{xxx} = 0. \tag{3}$$

Equation (2) can thus be considered a generalization of (3) to higher dimensions; it shares some of the same conservation laws and also has features in common with hydrodynamics. The fact that (3) is the Euler-Arnold equation of $\text{Diff}(S^1)$ with right-invariant H^1 metric is due to Misiołek [16] and Kouranbaeva [13].

Equation (1) arises in the case where $n = 1$ and $M = \mathbb{R}^3$ (viewed as the Heisenberg group) with the "standard" contact form $\theta = dz - y\,dx$. Here $u = -f_y \partial_x + (f_x + y f_z) \partial_y + (f - y f_y) \partial_z$ and $\lambda = f_z$. If the Riemannian metric is $ds^2 = dx^2 + dy^2 + (dz - y\,dx)^2$, the natural left-invariant metric on the Heisenberg group which makes M a Sasakian manifold (see Boyer [3]), we will have $m = f - f_{yy} - (1 + y^2)f_{zz} - 2yf_{xz} - f_{xx}$. The ansatz

$$f(t, x, y, z) = zg(t, y), \tag{4}$$

gives $m = z(g - g_{yy})$, $u = -zg_y \partial_x + yg \partial_y + z(g - yg_y) \partial_z$, and $\lambda = zg$, and equation (2) reduces to (1). A similar ansatz in ideal hydrodynamics leads to the Proudman-Johnson equation, which has been studied in [6, 17, 19, 9, 18, 5, 11]. The Proudman-Johnson equation was originally derived from the incompressible Euler equations by considering velocity fields of the form $\boldsymbol{u}(t, x, y) = (f(t, x), -yf_x(t, x))$, also known as stagnation-point similitude, which arise from a stream function $\psi(t, x, y) = yf(t, x)$ on an infinitely long 2D channel $(x, y) \in [0, L] \times \mathbb{R}$.

The outline of the paper is as follows. In §2 we establish some conservation laws and a local existence result for (1); the results of [10] do not apply here since our M is not compact, so we give an independent proof. In §3 we prove global existence of solutions to (1) for a class of initial data satisfying a particular sign condition. Finally in §4 we demonstrate the existence of solutions of (1) which blow up in finite time from smooth initial data.

2. **Local Existence.** First we derive some preliminary results. Set

$$\phi(t, y) = g(t, y) - g_{yy}(t, y) \tag{5}$$

and

$$\phi_0(y) = \phi(0, y) = g_0(y) - g_0''(y), \qquad g_0(y) = g(0, y). \tag{6}$$

Analogous to the Camassa-Holm equation, we may refer to (5) as the momentum associated to the velocity g. In terms of the momentum, equation (1) may be rewritten in the form

$$\phi_t + yg\phi_y = \left(yg_y - 4g\right)\phi. \tag{7}$$

Note that we may determine g from ϕ using the explicit solution formula

$$g(t, y) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-|y-y'|} \phi(t, y')\,dy'. \tag{8}$$

The characteristics are given by the solution of the flow equation

$$\frac{\partial \gamma}{\partial t}(t, y) = \gamma(t, y)g(t, \gamma(t, y)), \qquad \gamma(0, y) = y, \tag{9}$$

and in terms of the flow γ we obtain a convenient formula for the momentum conservation law. This formula should be considered analogous to the momentum transport law for Camassa-Holm and for the vorticity conservation law for the Euler equation of ideal hydrodynamics, in the sense that they all express the Noetherian conservation law arising from right-invariance of a Riemannian metric on a diffeomorphism group; see for example Arnold-Khesin [1].

Proposition 1. *If ϕ satisfies* (7) *and γ satisfies* (9)*, then we have*

$$\phi(t, \gamma(t, y)) = \frac{\phi_0(y) y^5 \gamma_y(t, y)}{\gamma(t, y)^5}. \tag{10}$$

Proof. Using the chain rule, we have

$$
\begin{aligned}
\frac{\partial}{\partial t} \log \phi(t, \gamma(t, y)) &= \frac{\phi_t(t, \gamma(t, y)) + \gamma(t, y) g(t, \gamma(t, y)) \phi_y(t, \gamma(t, y))}{\phi(t, \gamma(t, y))} \\
&= \gamma(t, y) g_y(t, \gamma(t, y)) - 4 g(t, \gamma(t, y)),
\end{aligned} \tag{11}
$$

using (7). Differentiating (9) with respect to y we have

$$\gamma_{ty}(t, y) = \gamma_y(t, y) g(t, \gamma(t, y)) + \gamma(t, y) \gamma_y(t, y) g_y(t, \gamma(t, y)), \tag{12}$$

which we can use to eliminate both g and g_y in (11). We obtain

$$\frac{\partial}{\partial t} \log \phi(t, \gamma(t, y)) = \frac{\gamma_{ty}(t, y)}{\gamma_y(t, y)} - 5 \frac{\gamma_t(t, y)}{\gamma(t, y)},$$

which can be easily integrated to obtain (10). □

A simple consequence of Proposition 1 is the conservation of the sign of the momentum, which is important for our global existence results.

Lemma 2.1. *Suppose* (1) *has a solution on $[0, T)$ for some $T > 0$. Then the flow $\gamma(t, y)$ is a strictly increasing diffeomorphism of $\mathbb{R}$ with $\gamma(t, 0) = 0$ for all $t \in [0, T)$. Furthermore if $\phi_0(y) \geq 0$ for all $y \in \mathbb{R}$, then $\phi(t, y) \geq 0$ and $g(t, y) \geq 0$ for all $t \in [0, T)$ and $y \in \mathbb{R}$. Similarly if $\phi_0(y) \leq 0$ for all $y \in \mathbb{R}$, then $\phi(t, y)$ and $g(t, y)$ are nonpositive.*

Proof. From equation (12) we see that

$$\gamma_y(t, y) = \exp\left(\int_0^t g(\tau, \gamma(\tau, y)) + \gamma(\tau, y) g_y(\tau, \gamma(\tau, y)) \, d\tau \right), \tag{13}$$

so that $\gamma_y(t, y) > 0$ for all t and y. Since $\gamma(0, 0) = 0$ we obviously have $\gamma(t, 0) = 0$ for all time, and we conclude that $\gamma(t, y) > 0$ if $y > 0$ and $\gamma(t, y) < 0$ if $y < 0$. Formula (10) then implies that $\phi(t, \gamma(t, y))$ has the same sign as $\phi_0(y)$ for every $y \in \mathbb{R}$. If $\phi(t, y) \geq 0$ for all $y \in \mathbb{R}$, the explicit solution formula (8) shows that $g(t, y) \geq 0$ as well. □

Another consequence of Proposition 1 is the local existence theorem, which we establish by writing everything in terms of γ as a "particle trajectory equation" and using Picard iteration, as in Chapter 4 of Majda-Bertozzi [14].

Theorem 2.2. *Suppose $g_0 \colon \mathbb{R} \to \mathbb{R}$ is a C^2 function such that $\phi_0 = g_0 - g_0''$ satisfies*

$$\sup_{y \in \mathbb{R}} y^2 |\phi_0(y)| \leq M \quad \text{for some } M. \tag{14}$$

Then there is a unique solution g of equation (1) *defined on $[0, T) \times \mathbb{R}$ for some $T > 0$ such that $g(t, y)$ is C^2 in y for each $t \in [0, T)$.*

Proof. By formula (8) we have

$$g(t, \gamma(t, y)) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-|\gamma(t,y)-y'|} \phi(t, y') \, dy' = \frac{1}{2} \int_{-\infty}^{\infty} e^{-|\gamma(t,y)-\gamma(t,z)|} \phi(t, \gamma(t, z)) \gamma_z(t, z) \, dz. \quad (15)$$

Using formula (10) and plugging this into (9), we obtain the differential equation

$$\frac{\partial \gamma}{\partial t}(t, y) = \frac{1}{2} \gamma(t, y) \int_{-\infty}^{\infty} e^{-|\gamma(t,y)-\gamma(t,z)|} \phi_0(z) \left[\frac{z}{\gamma(t, z)} \right]^5 \gamma_z(t, z)^2 \, dz. \quad (16)$$

We now view this as the equation

$$\frac{d\gamma}{dt} = F(\gamma(t)), \qquad \gamma(0) = y \mapsto y \quad (17)$$

on a certain open subset of a Banach space, where the function F is given by

$$F(\gamma) = y \mapsto \frac{1}{2} \gamma(y) \int_{-\infty}^{\infty} e^{-|\gamma(y)-\gamma(z)|} \phi_0(z) \left[\frac{z}{\gamma(z)} \right]^5 \gamma'(z)^2 \, dz. \quad (18)$$

Define a Banach space B by

$$B = \left\{ \gamma \colon \mathbb{R} \to \mathbb{R} \,\middle|\, \gamma(0) = 0 \text{ and } \sup_{y \in \mathbb{R}} |\gamma'(y)| < \infty \right\},$$

with norm $\|\gamma\| = \sup_{y \in \mathbb{R}} |\gamma'(y)|$. For numbers a and b satisfying $0 < a < 1 < b$, let U denote the open subset $U = \{\gamma \in B \,|\, a < \gamma'(y) < b \,\forall\, y \in \mathbb{R}\}$. Clearly U contains the identity, which is the initial condition for (17). Our goal is to show that F is Lipschitz on U, and we do this by showing that F has a uniformly bounded derivative on U.

If $v \in B$, we easily compute that

$$[DF_\gamma(v)](y) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-|\gamma(y)-\gamma(z)|} \zeta(z) \left[v(y) - \frac{5v(z)}{\gamma(z)} + \frac{2v'(z)}{\gamma'(z)} + \gamma(y) \operatorname{sgn}(y-z)[v(z) - v(y)] \right] dz. \quad (19)$$

where

$$\zeta(z) = \phi_0(z)[z/\gamma(z)]^5 \gamma'(z)^2. \quad (20)$$

Writing $w(y) = [DF_\gamma(v)](y)$, we just need to show that $|w'(y)|$ is bounded. The computation of w' is tedious but straightforward. Using $|v'(y)| \le c$ and $a \le \gamma'(y) \le b$, we obtain the estimate

$$|w'(y)| \le \frac{b^2 c}{2a^6} \int_{-\infty}^{\infty} e^{-a|y-z|} \phi_0(z) [a(1 + 3b|y| + b^2 y^2) + (1 + b|y|)(ab|z| + 5b + 2a)] \, dz.$$

Now by assumption we have $|\phi_0(z)| \le M'/(1 + |z|)^2$ for some constant M', and so we have

$$|w'(y)| \le C_1 (1 + |y|)^2 \int_{-\infty}^{\infty} \frac{e^{-a|y-z|} \, dz}{(1 + |z|)^2} + C_2 (1 + |y|) \int_{-\infty}^{\infty} \frac{e^{-a|y-z|} \, dz}{1 + |z|}. \quad (21)$$

The right side of (21) is bounded, for we can break it up into terms that have finite limits as $|y| \to \infty$, as follows:

$$\lim_{|y| \to \infty} \frac{\int_{-\infty}^{y} e^{az} \psi(z) \, dz}{e^{ay} \psi(y)} = \lim_{|y| \to \infty} \frac{e^{ay} \psi(y)}{e^{ay}(\psi'(y) + a\psi(y))} = \frac{1}{a}$$

by L'Hopital's rule since $\psi'(y)/\psi(y) \to 0$ for $\psi(y) = (1 + |y|)^{-k}$ when $k = 1$ or $k = 2$. Hence DF_γ is a bounded linear operator in B whenever $\gamma \in U$, and thus F is Lipschitz on U by the Mean Value Theorem. By Picard's Theorem [12], there is a unique solution for possibly short time with $\gamma(0)$ the identity.   □

Next we establish the non-existence of breaking wave solutions to (1) for initial data g_0 such that $\phi_0 \not\equiv 0$ does not change sign. Recall that the wave breaking phenomenon for a nonlinear wave equation is the existence of a blowup time T such that $|g_y(t, y_*)| \to \infty$ as $t \to T$ for some y_* while $|g(t, y_*)|$ remains bounded.

Lemma 2.3. *Let $T > 0$ denote the maximal life-span of g. If $g_0(y)$ is such that $\phi_0(y) = g_0(y) - g_0''(y)$ does not change sign, then $|g_y(t, y)| \le |g(t, y)|$ for all $t \in [0, T)$ and $y \in \mathbb{R}$. Hence no singularity in the form of wave breaking can occur on $(0, T]$.*

Proof. Writing formula (8) in the form

$$g(t, y) = \tfrac{1}{2} e^{-y} \int_{-\infty}^{y} e^{y'} \phi(t, y') \, dy' + \tfrac{1}{2} e^{y} \int_{y}^{\infty} e^{-y'} \phi(t, y') \, dy',$$

it is easy to check that

$$g(t, y)^2 - g_y(t, y)^2 = \left(\int_{-\infty}^{y} e^{y'} \phi(t, y') \, dy' \right) \left(\int_{y}^{\infty} e^{-y'} \phi(t, y') \, dy' \right). \tag{22}$$

As a result, if ϕ_0 never changes sign, then by Lemma 2.1 we know that ϕ is either non-negative or nonpositive for all $(t, y) \in [0, T) \times \mathbb{R}$. Hence the right side of (22) is nonnegative and we have the bound $|g_y(t, y)| \le |g(t, y)|$ for all y. $\qquad\square$

For the Camassa-Holm equation (3), it is known [7, 15] that if the initial momentum does not change sign, then the solution of the equation is global in time. For equation (1), we will see that even when the sign of the momentum is assumed constant, the behavior may be very different depending on whether it is positive or negative.

3. **Global Existence.** In this section we study global existence of certain solutions to (1). Theorem 3.1 below establishes global existence in time of solutions to (1) arising from initial data g_0 such that ϕ_0 is nonnegative.

Proposition 2. *Suppose g is a solution of (1) with initial condition ϕ_0 satisfying the decay condition (14). Then*

$$\int_{\mathbb{R}} \phi(t, y) \, dy \le \int_{\mathbb{R}} \phi_0(y) \, dy, \qquad t \in [0, T). \tag{23}$$

Proof. We first observe that by equation (10), if ϕ_0 satisfies the decay condition then so does $\phi(t, y)$ for any y. Using (5) we may write (1) as

$$\phi_t + 4g\phi = y\partial_y \left(gg_{yy} - g_y^2 \right). \tag{24}$$

Integrating the right side of (24) and using the decay condition to eliminate boundary terms, we obtain

$$\int_{\mathbb{R}} y\partial_y \left(gg_{yy} - g_y^2 \right) dy = 2 \int_{\mathbb{R}} g_y^2 \, dy, \tag{25}$$

which then yields

$$\frac{d}{dt} \int_{\mathbb{R}} g \, dy = -2 \left(2 \int_{\mathbb{R}} g^2 \, dy + \int_{\mathbb{R}} g_y^2 \, dy \right) < 0. \tag{26}$$

The result follows using the fact that $\int_{\mathbb{R}} \phi \, dy = \int_{\mathbb{R}} g \, dy$. $\qquad\square$

Amongst other conserved quantities, the integral of the momentum ϕ associated to the Camassa-Holm equation is known to be conserved [7]. The same is true in general for equation (2) on a compact manifold, but not in this case since the equation really "lives" on $\mathbb{R}^3$. Essentially what is happening is that the solutions of the form (4) are not well-behaved since for example they have infinite energy. The same type of phenomenon appears when solving the Euler equation of ideal hydrodynamics: in two dimensions finite-energy solutions exist globally [2], but there are infinite-energy solutions of the form (4) which can blow up in finite time. See for instance [6, 18] and references therein.

Theorem 3.1. *Suppose ϕ_0 satisfies the decay condition* (14) *and $\phi_0(y) \geq 0$ for all $y \in \mathbb{R}$. Then the solution g of* (1) *with initial data g_0 exists globally in time.*

Proof. By the general theory of ODEs in Banach spaces (see e.g. [12]), the only way a solution of $\gamma'(t) = F(\gamma(t))$ can blow up in finite time is if it leaves all of the bounded sets on which F is Lipschitz. Recall that we established the Lipschitz property under the condition that $a \leq \gamma_y(t, y) \leq b$ for some constants a and b satisfying $0 < a < 1 < b$, so we will have global existence as long as $\gamma_y(t, y)$ does not approach either zero or infinity in finite time.

From (13) we see that a uniform bound on both g and g_y is sufficient to control γ_y. By Lemma 2.1 we know that $g(t, y) \geq 0$ and $\phi(t, y) \geq 0$ for all t and y. Using this and the decay condition (14) to ensure that ϕ_0 is in L^1, (23) implies that

$$\|\phi(t)\|_{L^1} \leq \|\phi_0\|_{L^1} < \infty, \qquad t \in [0, T). \tag{27}$$

Using (8) we obtain

$$|g(t, y)| \leq \tfrac{1}{2} \int_{-\infty}^{\infty} |\phi(t, y)| \, dy = \tfrac{1}{2} \|\phi(t)\|_{L^1} \leq \tfrac{1}{2} \|\phi_0\|_{L^1}, \tag{28}$$

from which we conclude by Lemma 2.3 that g_y is uniformly bounded, and thus no blowup can occur. $\qquad\qquad\square$

4. **Blowup.** Constantin-Escher [7, 8] showed that solutions of the Camassa-Holm equation (3) cannot persist globally in time if the initial data f_0 is odd and satisfies $f_0'(0) < 0$. Further, they derived an upper bound, $T(f_0) = 1/2|f_0'(0)|$, for the maximal time of existence of solutions. Theorem 4.1 below establishes the existence of solutions to (1) which blow up in finite time from initial data g_0 that is both symmetric about $y = 0$ and satisfies $g_0(0) < 0$. Although singularities may form in (1) from nonpositive initial data[1], we note that solutions of (1) seem to retain a few properties that are inherent to the blowup mechanism of (3). For instance, an upper bound for the maximal time of existence of a solution to (1) is, analogous to that of (3), given by $1/\sqrt{6}|g_0(0)|$. Here $g_0(0) < 0$ serves as analogue to the Camassa-Holm condition $f_0'(0) < 0$ and blowup, in both cases, is to negative infinity. Moreover, in [8] it was shown that if the Camassa-Holm initial profile f_0 is even instead of odd, with $f_0'(0)$ negative enough, then $f_x(t, 0)$ can still diverge to negative infinity. A main difference between the qualitative behavior of blowup solutions to (1) and (3) is established in the second part of Theorem 4.1. More particularly, since (1) preserves the symmetry of the initial condition, for ϕ_0 both symmetric and nonpositive we show that solutions of (1) will actually diverge everywhere on $\mathbb{R}$.

Theorem 4.1. *Suppose g is a solution of* (1) *with initial condition $\phi_0 = g_0 - g_0''$ satisfying the decay condition* (14). *Furthermore, assume g_0 is even through $y = 0$ and $g_0(0) < 0$. Then $g(t, 0) \to -\infty$ as $t \nearrow T \leq 1/\left(\sqrt{6}|g_0(0)|\right)$. Additionally, if g_0 is such that $\phi_0 \leq 0$, then as $t \nearrow T$ we have $g(t, y) \to -\infty$ for all $y \in \mathbb{R}$.*

---

[1] A blowup feature that solutions to equations (1) and (3) do not share.

Proof. Suppose g_0 is even through $y = 0$ and $g_0(0) < 0$; let $T > 0$ denote the maximal life-span of g. Observe that (1) may be written as

$$g_t(t, y) = -g(t, y)^2 - yg(t, y)g_y(t, y) - p * \left[4g_y^2 + 3g^2 + y\partial_y\left(2g_y^2 - \frac{1}{2}g^2\right)\right] \qquad (29)$$

for $p(y) = \frac{1}{2}e^{-|y|}$. Integrating the last term in the bracket by parts, setting $y = 0$, and using symmetry of g about $y = 0$ then yields

$$g_t(t, 0) = -g(t, 0)^2 - \frac{1}{2}\int_0^\infty \left(4g_s^2 + 7g^2 + 4sg_s^2 - sg^2\right)e^{-s}ds \qquad (30)$$

for all $t \in (0, T)$. Now set

$$g(t, y) = e^{y/2}h(t, y) \qquad \text{for} \quad y \geq 0 \qquad (31)$$

and note that vanishing of $ye^{-y}g(t, y)^2$ as $y \to \infty$ implies the same for $yh(t, y)^2$. Using (31) on (30) we obtain

$$g_t(t, 0) = -g(t, 0)^2 - 2\int_0^\infty \left(2h^2 + h_s^2 + sh_s^2 + hh_s + shh_s\right)ds. \qquad (32)$$

Integrating the last two terms in (32) by parts and using the above decay condition of h now implies

$$\begin{aligned} g_t(t, 0) &= -2\int_0^\infty h_s^2\,ds - 3\int_0^\infty h^2\,ds - 2\int_0^\infty sh_s^2\,ds \\ &\leq 2\sqrt{6}\int_0^\infty hh_s\,ds - 2\int_0^\infty sh_s^2\,ds \\ &\leq -\sqrt{6}g(t, 0)^2, \end{aligned} \qquad (33)$$

from which we conclude that $g(t, 0) \to -\infty$ as $t \nearrow T \leq 1/(\sqrt{6}|g_0(0)|)$.

Now suppose $\phi_0(y) \leq 0$. Then Lemma 2.1 and the bound $|g_y(t, y)| \leq |g(t, y)|$, which was established in the proof of Lemma 2.3, imply that

$$|g(t, 0)|e^{-y} \leq |g(t, y)| \leq |g(t, 0)|e^y, \qquad y \geq 0, \quad t \in [0, T). \qquad (34)$$

Letting $t \nearrow T$ in (34) we see that $g(t, y) \to -\infty$ for all $y \geq 0$. Symmetry of g about $y = 0$ then yields our result for all $y \in \mathbb{R}$.

$\square$

REFERENCES

[1] V.I. Arnold and B. Khesin, *Topological Methods in Hydrodynamics*, Springer, New York 1998.

[2] C. Bardos, Existence et unicité de la solution de l'équation d'Euler en dimension deux, *J. Math. Anal. Appl.* **40** (1972), 769–790.

[3] Boyer, C.P., The Sasakian Geometry of the Heisenberg Group, *Bull. Math. Soc. Sci. Math. Roumanie* **52** (3), (2009), 251–262.

[4] R. Camassa and D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* **71** (11) (1993), 1661–1664.

[5] C. Cao, S. Ibrahim, K. Nakanishi and E.S. Titi, Finite-time blowup for the inviscid primitive equations of oceanic and atmospheric dynamics, arXiv:1210.7337.

[6] S. Childress, G.R. Ierley, E.A. Spiegel and W.R. Young, Blow-up of unsteady two-dimensional Euler and Navier-Stokes equations having stagnation-point form, *J. Fluid Mech.* **203** (1989), 1–22.

[7] A. Constantin and J. Escher, Global existence and blow-up for a shallow water equation, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **26** (2) (1998), 303–328.

[8] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.* **181** (2) (1998), 229–243.

[9] A. Constantin and M. Wunsch, On the inviscid Proudman-Johnson equation, *Proc. Japan Acad. Ser. A Math. Sci.*, 85, 7, (2009), 81-83.

[10] D.G. Ebin and S.C. Preston, Riemannian geometry of the contactomorphism group, in preparation.

[11] J. Escher and M. Wunsch, Restrictions on the geometry of the periodic vorticity equation, *Commun. Contemp. Math.* 14 (2012), no. 3, 1250016, 13 pp.

[12] P. Hartman, Ordinary differential equations, second edition, *SIAM*, Philadelphia (2002).

[13] S.O. Kouranbaeva, The Camassa-Holm equation as a geodesic flow on the diffeomorphism group, *J. Math. Phys.* **40** (2) (1999) 857–868.

[14] A. Majda and A. Bertozzi, Vorticity and incompressible flow, *Cambridge University Press*, Cambridge, (2002) 136–146.

[15] H.P. McKean, Breakdown of the Camassa-Holm equation, *Comm. Pure Appl. Math.* **57** (3) (2004) 416–418.

[16] G. Misiolek, A shallow water equation as a geodesic flow on the Bott-Virasoro group, *J. Geom. Phys.* **24** (3) (1998) 203–208.

[17] I. Proudman and K. Johnson, Boundary-layer growth near a rear stagnation point, *J. Fluid Mech.* **12** (2) (1962), 161–168.

[18] A. Sarria, Regularity of stagnation point-form solutions to the two-dimensional Euler equations, submitted, arXiv:1306:4756 (2013).

[19] R. Saxton and F. Tiglay, Global existence of some infinite energy solutions for a perfect incompressible fluid, *SIAM J. Math. Anal.* **40** (4) (2008), 1499–1515.

E-mail address: Stephen.Preston@colorado.edu
E-mail address: Alejandro.Sarria@colorado.edu