Geodesics, curvature, and conjugate points on Lie groups

Alice Le Brigant, Leandro Lichtenfelz, and Stephen C. Preston

Abstract

In a Lie group equipped with a left-invariant metric, we study the minimizing properties of geodesics through the presence of conjugate points. We give criteria for the existence of conjugate points along steady and nonsteady geodesics, using different strategies in each case. We consider both general Lie groups and quadratic Lie groups, where the metric in the Lie algebra $g(u, v) = \langle u, \Lambda v\rangle$ is defined from a bi-invariant bilinear form and a symmetric positive definite operator Λ. By way of illustration, we apply our criteria to $SO(n)$ equipped with a generalized version of the rigid body metric, and to Lie groups arising from Cheeger's deformation technique, which include Zeitlin's $SU(3)$ model of hydrodynamics on the 2-sphere. Along the way we obtain formulas for the Ricci curvatures in these examples, showing that conjugate points occur even in the presence of some negative curvature.

Contents

1 Introduction

One of the most basic questions in Riemannian geometry is the behavior of geodesics emanating from a point, and how it compares to the behavior in flat space. There is a unique unit-speed, length minimizing geodesic from any base point to a sufficiently nearby point, but for longer distances there may be multiple geodesics reaching the same point, or a geodesic may eventually fail to minimize length. More generally the exponential map, which takes an initial unit velocity vector to its geodesic endpoint, may fail to be locally invertible. This, roughly speaking, is what it means for a geodesic to have a conjugate point. This phenomenon has important topological implications, and is one of the main methods to demonstrate nontrivial topology in a manifold using only local quantities such as curvature.

As an example, the determination of the conjugate locus of a Riemannian manifold is a classical subject which goes back to at least Jacobi [30], whose famous "last geometric statement" says that the conjugate locus of a (non-umbilical) point on the two-dimensional ellipsoid has exactly four cusps. The fact that this seemingly simple conjecture remained open for almost 150 years, until being settled in 2004 by Itoh and Kiyohara [26], highlights how nontrivial the task of computing conjugate points can be, even in an explicit example with simple curvature formulas.

Unfortunately in many cases the curvatures are difficult to compute. Even when curvature is computable, for generic manifolds we typically do not have the sort of sign-definite results on curvature needed to use comparison theory to get strong results. The situation is greatly simplified if we have symmetry in the manifold, such as a transitive Lie group of isometries acting on it (the homogeneous case). The simplest case here is when the manifold itself is a Lie group, and all left-translations are isometries; equivalently one may require all right-translations to be isometries. If one requires both left- and right-translations to be isometries, then we have a bi-invariant metric, and the situation simplifies greatly—so much so that the topology is severely restricted and the sectional curvature is forced to be nonnegative. With only one-sided invariance, the curvature is typically still rather complicated and takes on both signs. Thus standard conjugate point results that require a positive lower bound on some curvature do not apply (this includes many results arising out of the original Sturm comparison theorem, leading to the Rauch comparison theorem and the Morse index theorem [13], and the Gromoll-Meyer theorem [21] on Ricci curvature comparison). The well-known paper of Milnor [37] describes a number of cases where the curvature signs (whether sectional, Ricci, or scalar) are well-determined and many when they are not.

Our goal in this paper is to describe results on the geodesics and conjugate points that can be derived using the left-invariant metric structure but do not rely directly on curvature. Instead we take advantage of the Noether principle that a large group of isometries leads to conserved quantities in the equations of motion, and in particular to simplifications. Thus for example the second-order geodesic equation splits into a decoupled system (the flow equation and the Euler-Arnold equation), and similarly the Jacobi equation needed to describe conjugate points also decouples. This allows us to demonstrate several results that illustrate under what circumstances we can say a geodesic must eventually experience a conjugate point on a Lie group, even if the curvature changes sign. Applications include any physical system where the configuration space consists of group elements which preserve the physics, including in particular the rigid body as the oldest example, where the geodesic equation arises from the Euler equation.

A basic dichotomy we emphasize is the difference between steady and nonsteady solutions of the Euler equation. Essentially, "steady" describes those geodesics under a left-invariant metric where the tangent vector is always the left-translation of the initial vector, while "nonsteady" describes all others. Equivalently, a steady solution of the Euler equation is an initial velocity vector where the Riemannian exponential map coincides with the group exponential map; this always happens in the bi-invariant case but is rare in the left- or right-invariant case. Roughly speaking, our results for nonsteady geodesics are given in terms of a natural orthogonal triad of vectors along the geodesic, while our results for steady geodesics are given in terms of algebraic properties. The latter technique is much more well-studied in manifolds with symmetry: for example in symmetric, normal homogenenous or naturally reductive spaces, this method was carried out quite effectively by Rauch, Chavel, Ziller et al ([42, 8, 9, 10, 46, 47]).

There are far fewer results on nonsteady solutions. In this paper, we will give one result on nonsteady solutions in the most general case of an arbitrary left-invariant metric on a Lie group, which shows that if the geodesic happens to be closed, then there must be a conjugate point (and not merely a cut point) along the geodesic. Thus for example compact nonpositive curvature manifolds which have many closed geodesics without conjugate points are quite different from Lie groups with left-invariant metrics (and only the flat torus is an example of both).

To obtain more information, we impose more structure. As mentioned, the bi-invariant case is essentially trivial. However the case of quadratic Lie groups (where there is a nondegenerate bi-invariant quadratic form, and a Riemannian metric is generated from this via a symmetric operator from the Lie algebra to itself) is nontrivial but still allows for a number of fairly general statements. There are two well-known examples of this situation: first, the (generalized) rigid body $SO(n)$, i.e., the group of all rotations of $\mathbb{R}^n$ with the symmetric operator generated by the moments of inertia of the body. Second, the Zeitlin model for spherical hydrodynamics, which is described in terms of a metric on $SU(n)$ meant to approximate the motion of an ideal incompressible fluid on the sphere with a truncated model. We will discuss both examples in detail later in this paper. Another particularly interesting class of examples is given by Cheeger's deformation technique, which perturbs a bi-invariant metric on a Lie group by a multiple of the metric on a subgroup, for example $SU(n)$ perturbed by $SO(n)$ (which reduces in the case $n = 2$ to the Berger spheres); in this case we will describe several new results on the geodesics, conjugate points and Ricci curvature. Details are given in the next section.

1.1 Main results

Let G be a Lie group equipped with a left-invariant metric g. We are interested in its geodesics, which in many examples describe the evolution of some underlying physical phenomenon (the movement of rigid bodies or fluids). By left-invariance, geodesics can be translated to start at the identity, and there they are solutions of the following system of equations (see Proposition 2.1 in Section 2), respectively the flow and Euler-Arnold equations,

$$\gamma'(t) = \gamma(t)u(t), \qquad u'(t) = \mathrm{ad}^{\star}_{u(t)}u(t), \qquad \gamma(0) = \mathrm{id}, u(0) = u_0. \tag{1.1}$$

Here the operator $\mathrm{ad}^{\star}$ is defined by the condition

$$g(\mathrm{ad}^{\star}_u v, w) = g(v, \mathrm{ad}_u w) \qquad \forall u, v, w \in \mathfrak{g}.$$

We will separately consider the steady case, when $u(t) = u_0$ for all t, and the nonsteady case, when u' is not identically zero, and in fact never zero, by uniqueness of solutions of (1.1).

Our first result implies that all nonsteady, closed geodesics in any Lie group G with a left-invariant metric contain conjugate points.

Theorem 1.1. *Suppose $\gamma(t)$ is a solution of (1.1), with velocity $u(t)$ nonconstant. If right multiplication by $\gamma(\tau)\gamma(0)^{-1}$ is an isometry of the left-invariant metric for some $\tau > 0$, then $\gamma(\tau)$ is conjugate to $\gamma(0)$.*

Note that on a general manifold there may be many closed geodesics which have no conjugate points, e.g., on a compact manifold with nonpositive curvature. In the context of Lie groups, a steady closed geodesic can also be free of conjugate points, as for example on the flat torus. Obviously if a geodesic is closed, it is no longer uniquely minimizing up to its midpoint (since two distinct geodesics connect its starting point and its midpoint), and in this situation we have a cut point; however not every cut point is a conjugate point, since conjugate points describe continuous families of geodesics that all infinitesimally reach the same point.

As a consequence of Theorem 1.1, if a Lie group with left-invariant metric has a closed geodesic, either it arises from a steady solution of the Euler equation, or it has some positive sectional curvature along it. This gives an alternate proof of the known fact that the only compact Lie group with nonpositive sectional curvature is the abelian flat torus (which arises from the fact that nonpositive curvature requires solvability of the group, and the only compact solvable groups are abelian – see, e.g., [31]).

In the case of a quadratic Lie group, we observe the following phenomenon: along any nonsteady solution $u(t)$, there exists a special 3-frame of orthogonal vector fields along the geodesic in which the index form takes a particularly simple form. On a three-dimensional Lie group, this would give a complete description of conjugate points, but even on higher-dimensional groups, energy perturbations in this specific frame seem to capture "most" of the saddle point behavior in the index form which characterizes conjugate points.

Roughly speaking, this basis is constructed using the velocity field $u(t)$, its derivative $u'(t)$, and a version of their cross product. This construction relies on conserved quantities which are direct analogues of the conservation laws for energy and angular momentum in the case of a rigid body, but exist on any quadratic Lie group. Surprisingly, this sometimes makes the nonsteady case easier than the steady case. Note that most results on conjugate points on Lie groups have been shown for steady solutions of the Euler equation (1.1) (cf. [14] and references therein).

We give an informal statement of the next theorem, postponing the details of the construction and the proof to Chapter 3. It shows that one can compute the index form without a curvature formula, and that bounds on a particular scalar function of time along the geodesic can give a sufficient criterion for conjugate points. While the details are somewhat complicated in general, in particular cases the criterion is quite easily computable, and shows for example that even on certain Lie groups with indefinite Ricci curvature, we can still guarantee existence of conjugate points.

Theorem 1.2. *Let G be a quadratic Lie group with a left-invariant metric g given by* (2.12) *for an operator Λ with a bi-invariant nondegenerate bilinear form $\langle\cdot,\cdot\rangle$. Let $u(t)$ be a non-steady solution of the Euler equation. Then, the quantities $k := g(u, u)$ and $\ell := g(u, \Lambda u)$ are conserved, and the vectors $\{u, u', k\Lambda u - \ell u\}$ form a mutually orthogonal basis. Furthermore, there is a scalar function $\psi(t)$, depending only on this basis and the operator Λ, such that if $\psi(t)$ is bounded from below by a positive constant, then there is eventually a conjugate point along the geodesic $\gamma(t)$ corresponding to $u(t)$ via* (1.1).

Remark 1.3. *A weaker criterion, which is still applicable in many examples, is available in the full statement of Theorem 1.2, as we shall see in Section* 3.

In the steady case, one must use a different approach, since there is no natural frame along the geodesic. We directly study the Jacobi equation to obtain a criterion for the existence of conjugate points along any steady geodesic on a general Lie group. In some cases, this gives a necessary and sufficient condition.

Theorem 1.4. *Suppose u_0 is a steady solution of the Euler-Arnold equation* (1.1)*, i.e., that* $\mathrm{ad}^\star_{u_0} u_0 = 0$*. Let L and F be the linear operators on the Lie algebra $\mathfrak{g}$ defined by*

$$L(v) := \mathrm{ad}_{u_0} v, \qquad F(v) = \mathrm{ad}^\star_{u_0} v + \mathrm{ad}^\star_v u_0.$$

Then both operators map the g-orthogonal complement of u_0 to itself. If there is an operator R defined on this orthogonal complement such that $RF + LR = I$, then there is a conjugate point along the geodesic if and only if for some $\tau > 0$ we have

$$\det\left(e^{\tau L} R e^{\tau F} - e^{-\tau L} R e^{-\tau F}\right) = 0.$$

The criterion of Theorem 4.1 is particularly simple in the case of quadratic Lie groups, where it is a consequence of a commutativity condition.

Theorem 1.5. *Suppose G is a quadratic group, with a left-invariant metric g given by* (2.12) *for an operator Λ with a bi-invariant nondegenerate bilinear form $\langle\cdot,\cdot\rangle$. Let u_0 be an eigenvector of Λ with $\Lambda u_0 = \lambda u_0$, and let $L = \mathrm{ad}_{u_0}$ as before. If L^2 commutes with Λ, then there is eventually a conjugate point along the geodesic $\gamma(t)$ solving the geodesic equation* (2.14).

We will demonstrate how these criteria apply in two main examples. First, on $G = SO(n)$, we show that when the metric comes from a higher-dimensional rigid body, the Ricci curvatures are always positive, which guarantees conjugate points by a result of Gromoll and Meyer. However, even for more general metrics on $SO(n)$ where the Ricci curvature can be negative, we show that the criterion of Theorem 4.2 guarantees conjugate points, and there are conjugate points along *every* steady geodesic arising from an eigenvector of Λ, regardless of whether it is stable or unstable in the Eulerian sense.

The second class of examples is what we call here *Berger-Cheeger* groups. They are a family of quadratic Lie groups where the left-invariant metric is induced by the operator $\Lambda = I + \delta P$, where $\delta \in \mathbb{R}$ and P is the orthogonal projection onto $\mathfrak{h}$, the Lie algebra of a subgroup H of G. The simplest example is when $G = SU(n)$ and $H = SO(n)$ with their usual bi-invariant metrics. In particular, when $n = 3$, H is the so-called "spin representation" of $SO(3)$ in $SU(3)$, and for $\delta = -\frac{2}{3}$, we obtain the Zeitlin model for hydrodynamics of the two-sphere [39]. We demonstrate how to solve the Euler-Arnold equation to obtain an explicit formula for every geodesic. In addition we obtain a simple block-diagonal form for the Ricci curvature, showing that it can easily be made either strictly positive or of mixed sign depending on the parameter δ. We apply the criterion of Theorem 1.2, thus showing that despite the changes in curvature as δ varies, the behavior of geodesics and conjugate points remains essentially the same.

In summary, the techniques we present here show that Lie groups with one-sided invariant metrics, and particularly quadratic Lie groups, have many conjugate points which can be detected by a variety of methods. The description of the conjugate locus, and the implications on the topological properties of the Lie group, would be an interesting set of topics to study in the future.

1.2 Outline of the paper

In Section 2, we provide background material on geodesics and Jacobi fields on left-invariant metric on Lie groups, with a specific focus on quadratic Lie groups. In Section 3, we prove Theorem 1.1 for nonsteady closed geodesics, and Theorem 1.2 for any nonsteady geodesic in the quadratic case. Section 4 is devoted to the proofs of Theorems 4.1 and 4.2 in the steady case. Finally, we consider the case of $SO(n)$ for generalized rigid bodies in Section 5 and we study Berger-Cheeger groups in Section 6.

2 Background

2.1 General left-invariant metrics

We first summarize some basic facts about general left-invariant metrics on Lie groups. We refer the reader to do Carmo [13] for further details concerning standard Riemannian geometry facts, Milnor [37] for curvature and group-theoretic properties, Arnold-Khesin [3] for stability and properties of geodesics, and Khesin et al. [25] for information about the Jacobi equation and conjugate points.

Let G be any finite-dimensional Lie group, and g an inner product on its Lie algebra $\mathfrak{g}$. It defines a left-invariant Riemannian metric on G in the following way: for any $\eta \in G$ and $x, y \in T_pG$,

$$g_\eta(x, y) := g(u, v), \quad \text{where} \quad x = \eta u,\ y = \eta v,\ u, v \in \mathfrak{g}.$$

Let $\gamma(t)$, $t \in [0,1]$, be a curve in G and $u(t) = \gamma(t)^{-1}\gamma'(t)$ its velocity vector left translated to the Lie algebra. Then γ is a geodesic for g if and only if it is a critical point of the energy functional

$$E(\gamma) := \frac{1}{2}\int_0^1 g(\gamma', \gamma')dt = \frac{1}{2}\int_0^1 g(u,u)dt,$$

where we have used the left-invariance of g. From this characterization, we can write the geodesic equations purely in terms of $u(t)$ and the operator $\mathrm{ad}^\star$ defined by the condition

$$g(\mathrm{ad}_u^\star v, w) = g(v, \mathrm{ad}_u w) \qquad \forall u, v, w \in \mathfrak{g}. \tag{2.1}$$

We provide a short variational argument for the following well known fact.

Proposition 2.1 ([3])**.** *Suppose G is a Lie group with left-invariant metric g. A curve $\gamma : [0,1] \to G$ is a geodesic for g if and only if it satisfies*

$$\gamma'(t) = \gamma(t)u(t), \qquad u'(t) = \mathrm{ad}_{u(t)}^\star u(t). \tag{2.2}$$

Proof. For $\epsilon > 0$ and $(t,s) \in [0,1] \times (-\epsilon, \epsilon)$, let $\eta : (s,t)$ be a variation of γ with fixed end points:

$$\eta(0,t) = \gamma(t), \quad \eta(s,0) = \gamma(0), \quad \eta(s,1) = \gamma(1).$$

We translate the partial derivatives of η to the Lie algebra to define

$$u(s,t) = \eta(s,t)^{-1}\partial_t\eta(s,t), \quad v(s,t) = \eta(s,t)^{-1}\partial_s\eta(s,t),$$

The derivative of the energy functional with respect to s is

$$\frac{\partial}{\partial s}E(\eta) = \int g(\partial_s u, u)dt = \int (\mathrm{ad}_u v + \partial_t v, u)dt = \int g(v, \mathrm{ad}_u^* u - \partial_t u)dt, \tag{2.3}$$

where we have used integration by parts and the zero-curvature formula [37]

$$\partial_s u - \partial_t v = \mathrm{ad}_u v$$

The curve γ is a critical point of the energy if and only if the derivative (2.3) vanishes at $s = 0$ for any value of $\partial_s\eta(0,\cdot)$, or equivalently any $v(0,\cdot)$, which is equivalent to $\mathrm{ad}_u^* u = \partial_t u$, as claimed. □

The following terminology, mentioned in the introduction, will be used throughout the paper, so we repeat it here in one definition.

Definition 2.2. *The first equation in* (2.2) *is called the* flow equation*, while the second is called the* Euler-Arnold *equation. When $\mathrm{ad}_{u_0}^\star u_0 = 0$, then $u(t) = u_0$ for all t, and u_0 is called a* steady *solution of the Euler-Arnold equation. If $\mathrm{ad}_{u_0}^\star u_0 \neq 0$, then $u'(t)$ is never zero, by uniqueness of solutions of ODEs, and in this case we call the solution $u(t)$* nonsteady*.*

Note that if u_0 is steady, then the corresponding geodesic $\gamma(t)$ is a one-parameter subgroup of G. The Euler-Arnold equation can also be rewritten as a conservation law.

Proposition 2.3 ([3])**.** *Suppose G is a Lie group with a left-invariant metric g. For $\eta \in G$, let $\mathrm{Ad}^\star_\eta \colon \mathfrak{g} \to \mathfrak{g}$ denote the operator defined by the condition*

$$g(\mathrm{Ad}^\star_\eta u, v) = g(u, \mathrm{Ad}_\eta v) = g(u, \eta v \eta^{-1}) \qquad \forall u, v \in \mathfrak{g}.$$

Then the geodesic equation (1.1) *may be written in the form*

$$\frac{d}{dt}\Big(\mathrm{Ad}^\star_{\gamma(t)^{-1}} u(t)\Big) = 0.$$

The Riemannian exponential map $\exp\colon \mathfrak{g} \to G$ based at the identity $T_{\mathrm{id}}G \simeq \mathfrak{g}$ sends each initial condition u_0 to $\gamma(1)$, where γ solves (2.2) with $\gamma(0) = \mathrm{id}$ and $\gamma'(0) = u_0$. By homogeneity, we have $\gamma(\tau) = \exp(\tau u_0)$. Its derivative

$$D(\exp)(\tau u_0) : \mathfrak{g} \to T_{\gamma(\tau)}G \tag{2.4}$$

is always an invertible linear map for small values of τ, depending on u_0, but can become singular for large τ. A *conjugate point* along a geodesic γ is defined to be a value $\gamma(\tau) = \exp(\tau u_0)$ for which the linear map (2.4) is singular. Geometrically, this means that there exists a family of geodesics containing γ that start at $\gamma(0)$, spread out and eventually reconverge, up to first order, at $\gamma(\tau)$.

Such a variation of the Riemannian exponential map defines a Jacobi field

$$J(t) = \frac{\partial}{\partial s}\bigg|_{s=0} \gamma(s, t).$$

satisfying $J(0) = 0$ and $J(\tau) = 0$. A Jacobi field verifies the Jacobi equation,

$$\frac{D^2 J}{dt^2} + R(J(t), \gamma'(t))\gamma'(t) = 0, \tag{2.5}$$

where R is the Riemann curvature tensor and $\frac{D}{dt}$ is the covariant derivative. This is obtained by linearizing the geodesic equation. On a Lie group, equation (2.5) can be translated to the Lie algebra using the group structure.

Proposition 2.4. *Let G be a Lie group with left-invariant metric g, $\gamma : [0, 1] \to G$ a geodesic on G, and $\gamma(s, t)$ a family of curves such that $\gamma(0, t) = \gamma(t)$ for all t. We define*

$$u(s, t) := \gamma(s, t)^{-1}\partial_t \gamma(0, t), \quad y(t) = \gamma(t)^{-1}\partial_s \gamma(0, t), \quad z(t) = \partial_s u(0, t). \tag{2.6}$$

Then $J(t) := \partial_s \gamma(0, t)$ is a Jacobi field along γ if and only if

$$y'(t) + \mathrm{ad}_{u(t)} y(t) = z(t), \qquad z'(t) = \mathrm{ad}^\star_{u(t)} z(t) + \mathrm{ad}^\star_{z(t)} u(t). \tag{2.7}$$

Proof. Let us extend the definitions of y and z to any (s, t) in the obvious manner. Then the first equation is just the zero curvature formula

$$z(t) - y'(t) = \partial_s u(0, t) - \partial_t y(0, t) = \mathrm{ad}_{u(0,t)} y(0, t).$$

The second equation directly results from differentiating the Euler-Arnold equation, i.e., the second equation in (1.1), with respect to s. □

Thus, having a nonzero $y(t)$ as in Proposition 2.4 satisfying $y(0) = y(\tau) = 0$ is equivalent to $\gamma(0)$ and $\gamma(\tau)$ being conjugate.

Conjugate points are naturally linked to the sign of curvature. Having sectional curvature bounded below by a positive constant along all sections containing the geodesic's tangent vector is sufficient to guarantee conjugate points along the geodesic by the Rauch comparison theorem [13]. In fact, having Ricci curvature bounded below by a positive constant along the geodesic is also sufficient by a result of Gromoll-Meyer [21]. However, on most Lie groups curvatures take both signs (see Milnor [37]), and comparison theory cannot be used in general to guarantee conjugate points.

The fact remains that the existence of a conjugate point requires there to be at least some positive sectional curvature along the geodesic. This can easily be seen by considering the index form, defined for vector fields $W(t)$ along $\gamma(t)$ that vanish at $t = 0$ and $t = \tau$ by

$$I(W, W) = \int_0^\tau g\Big(\frac{DW}{dt}, \frac{DW}{dt}\Big) - g\big(R\big(W(t), \gamma'(t)\big)\gamma'(t), W(t)\big)\, dt, \tag{2.8}$$

where R is the curvature tensor and $\frac{D}{dt}$ is the covariant derivative. A standard fact in Riemannian geometry [13] is that the index form is negative for some $W(t)$ vanishing at the endpoints if and only if there is a Jacobi field $J(t)$ which vanishes at time $t = 0$ and some at some time $t = t_0$ with $0 < t_0 < \tau$. In particular, there can be no conjugate points if the sectional curvature is everywhere non-positive, since the sectional curvature $K(W, \gamma')$ is proportional to $g(R(W, \gamma')\gamma', W)$. On a Lie group, this quantity can be computed by Arnold's formula [3]

$$g(R(u, v)v, u) = \frac{1}{4}\|\mathrm{ad}_u^\star v + \mathrm{ad}_v^\star u\|^2 - g(\mathrm{ad}_u^\star u, \mathrm{ad}_v^\star v) - \frac{3}{4}\|\mathrm{ad}_u v\|^2 + \frac{1}{2} g(\mathrm{ad}_u v, \mathrm{ad}_v^\star u - \mathrm{ad}_u^\star v),$$

for any $u, v \in \mathfrak{g}$. In the sequel, it will be helpful to use a slightly different formula, as in [32],

$$g(R(u, v)v, u) = \frac{1}{4}\|\mathrm{ad}_u^\star v + \mathrm{ad}_v^\star u + \mathrm{ad}_u v\|^2 - g(\mathrm{ad}_u^\star v + \mathrm{ad}_u v, \mathrm{ad}_u v) - g(\mathrm{ad}_u^\star u, \mathrm{ad}_v^\star v). \tag{2.9}$$

As for the index form (2.8), it can be written in the following way.

Proposition 2.5. *In the notations of Proposition 2.4, the index form is given by*

$$I(y, y) = \int_0^\tau g(z(t), z(t)) - g(\mathrm{ad}_{y(t)} z(t), u(t))\, dt, \tag{2.10}$$

for variations $y(t)$ which vanish at $t = 0$ and $t = \tau$. If there is such a y and $\tau > 0$ such that $I(y, y) < 0$, then there is a conjugate point occurring at a time $t = t_0 < \tau$.

Proof. As usual with second-order self-adjoint differential equations, the index form is obtained by multiplying the second Jacobi equation (2.7) by the negative of the variation field and integrating over $[0, \tau]$, using the vanishing at the endpoints to eliminate the boundary condition.

$$\begin{aligned} I(y, y) &= -\int_0^\tau g\big(y(t), -z'(t) + \mathrm{ad}_{u(t)}^\star z(t) + \mathrm{ad}_{z(t)}^\star u(t)\big)\, dt \\ &= \int_0^\tau g\big(y'(t), z(t)\big) + g\big(\mathrm{ad}_{u(t)} y(t), z(t)\big) + g\big(\mathrm{ad}_{z(t)} y(t), u(t)\big)\, dt \\ &= \int_0^\tau g\big(z(t), z(t)\big) - g\big(\mathrm{ad}_{y(t)} z(t), u(t)\big)\, dt \end{aligned}$$

□

A well-known method to find conjugate points that relies on the index form is the Misiołek criterion, particularly effective for finding conjugate points along steady flows of the Euler equation for ideal fluids [14, 38, 40], which can also be used for some nonsteady flows such as Rossby-Haurwitz waves [5]. Although it has so far only been applied in the context of geometric hydrodynamics, it can be readily generalized to any Lie group as follows.

Given an initial condition u_0 which determines a steady solution of the Euler-Arnold equation, let $y(t) = f(t)v$ be a variation vector field, for some fixed vector v and scalar function $f(t)$. Then $z(t) = f'(t)v + f(t)\mathrm{ad}_{u_0}v$, and the index form (2.10) becomes

$$\begin{aligned} I(y,y) &= \int_0^T (f')^2\|v\|^2 + 2ff'g\big(v, \mathrm{ad}_{u_0}v\big) + f^2\|\mathrm{ad}_{u_0}v\|^2 - f^2 g\big(\mathrm{ad}_v(\mathrm{ad}_{u_0}v), u_0\big)\,dt \\ &= \int_0^T (f')^2\|v\|^2 + 2ff'g\big(v, \mathrm{ad}_{u_0}v\big) + f^2 g\big(\mathrm{ad}_v u_0 + \mathrm{ad}_v^* u_0, \mathrm{ad}_v u_0\big)\,dt. \end{aligned}$$

Now choosing $f(t) = \sin\left(\frac{\pi t}{T}\right)$ and taking $T > 0$ large, the first term can be made arbitrarily small since $f' \sim 1/T$ and the second even integrates to zero, so if

$$g\big(\mathrm{ad}_v u_0 + \mathrm{ad}_v^* u_0, \mathrm{ad}_v u_0\big) < 0 \qquad \text{for some } v \in \mathfrak{g}, \tag{2.11}$$

then the geodesic with initial velocity u_0 will eventually develop conjugate points. Condition (2.11) is precisely the Misiołek criterion. Comparing it with formula (2.9), and using the fact that u_0 is a steady solution, we see that (2.11) directly implies positive curvature on the 2-plane spanned by u_0 and v.

In 5.2, we give new examples on generalized rigid bodies where the Misiołek criterion does not apply even in the steady case, but Theorem 4.2 is able to detect conjugate points.

2.2 Quadratic Lie groups

A special focus of this paper is on quadratic Lie groups. A Lie group G is said to be *quadratic* if there is a nondegenerate bilinear form $\langle\cdot,\cdot\rangle$ (not necessarily positive-definite) on the Lie algebra, which has the property of being bi-invariant

$$\langle \mathrm{ad}_u v, w\rangle + \langle v, \mathrm{ad}_u w\rangle = 0 \qquad \text{for all } u, v, w \in \mathfrak{g}.$$

Not every Lie group has such a bilinear form: compact and abelian groups have positive-definite ones [37], and more generally any semisimple Lie group has a nondegenerate Killing form which is bi-invariant [22], while for example the upper half-plane considered as a Lie group has none. Every left-invariant Riemannian metric on a quadratic group G can be defined from an invertible symmetric operator Λ on $\mathfrak{g}$

$$g(u,v) = \langle u, \Lambda v\rangle. \tag{2.12}$$

The first example is $SO(n)$ with the kinetic energy metric describing the rotations of a free multidimensional rigid body. The bi-invariant metric is $\langle u, v\rangle = -\frac{1}{2}\,\mathrm{Tr}(uv)$, and the operator $\Lambda\colon \mathfrak{so}(n) \to \mathfrak{so}(n)$ is given by $\Lambda(u) = \frac{1}{2}(Mu+uM)$, for a symmetric positive-definite matrix M determined by the moments of inertia. The Euler equations in this case (see [19]) are a completely integrable system [35], and the steady solutions and their stability have been studied by several authors (see [27] and references therein). We will say more about this in Section 5.

The second family of examples is given by Zeitlin's model of spherical hydrodynamics on $SU(n)$ ([39, 45]). Here one starts with matrices $s_1, s_2, s_3 \in \mathfrak{su}(n)$ coming from a representation of $\mathfrak{so}(3)$, so that they satisfy the cyclic commutation relations $[s_i, s_j] = s_k$ when (i, j, k) is a positive permutation of $(1, 2, 3)$. The Hoppe-Yau Laplacian [23] is then defined by $\Delta = \sum_i \mathrm{ad}^2_{s_i}$. With respect to the bi-invariant metric $\langle u, v\rangle = -\frac{1}{2}\,\mathrm{Tr}(uv)$, the operator Δ is symmetric and positive-definite, and its eigenvalues are given by $-\ell(\ell+1)$ (each with multiplicity $2\ell+1$) for $1 \leq \ell \leq n-1$. The corresponding Euler-Arnold equation with $\Lambda = -\Delta$ gives a surprisingly good [39] finite-dimensional approximation of two-dimensional ideal fluid dynamics on the sphere due to some convenient properties of spherical harmonics.

An interesting family of examples comes from the following deformation related to a construction of Cheeger [11]. Let G be a compact group and H a closed subgroup, with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Then G has a positive-definite bi-invariant metric, and there is an orthogonal projection $P\colon \mathfrak{g} \to \mathfrak{h}$. Defining $\Lambda = I + \delta P$ for a real parameter $\delta > -1$, we obtain a family of metrics that we call Berger-Cheeger groups, since when $G = S^3$ and $H = S^1$ these give the Berger spheres. We will give more details on this family in Section 6, but for now we note that when $G = SU(3)$ and $H = SO(3)$, the choice $\delta = -\frac{2}{3}$ gives exactly the Zeitlin model in the previous paragraph (since the only two eigenvalues of Λ in that case are 2 and 6).

The infinite-dimensional Lie groups arising in Arnold's well-known description of ideal fluid motion [2], i.e., the groups of volume-preserving diffeomorphisms of a compact manifold of two [16] or three [44] dimensions, also fit in the quadratic framework. Even though these infinite-dimensional Lie groups are beyond the scope of this paper, we hope that the methods we develop in this work can shed some light on their geometry.

In the quadratic setting, the operator $\mathrm{ad}^\star$ can be written in terms of Λ.

Lemma 2.6. *If $\langle\cdot,\cdot\rangle$ is a nondegenerate bi-invariant bilinear form on $\mathfrak{g}$, and if Λ is a symmetric operator such that $g(u, v) := \langle u, \Lambda v\rangle$ defines a Riemannian metric, then*

$$\mathrm{ad}^\star_u v = -\Lambda^{-1}(\mathrm{ad}_u \Lambda v). \tag{2.13}$$

Proof. Since $\langle\cdot,\cdot\rangle$ is nondegenerate, it is sufficient to verify that the condition in (2.1) holds for all $u, v, w \in \mathfrak{g}$: we have

$$g\big(-\Lambda^{-1}(\mathrm{ad}_u\Lambda v), w\big) = -\langle \mathrm{ad}_u\Lambda v, w\rangle = \langle \Lambda v, \mathrm{ad}_u w\rangle = g(v, \mathrm{ad}_u w),$$

as desired. Here we used bi-invariance of $\langle\cdot,\cdot\rangle$ in order to say that

$$-\langle \mathrm{ad}_u\Lambda v, w\rangle = \langle \Lambda v, \mathrm{ad}_u w\rangle.$$

□

We can now express the Euler-Arnold equation in the quadratic case.

Lemma 2.7. *If $\langle\cdot,\cdot\rangle$ is a bi-invariant bilinear form on $\mathfrak{g}$, and if Λ is a symmetric operator such that $g(u,v) := \langle u, \Lambda v\rangle$ defines a Riemannian metric, then the geodesic equation is given by*

$$\gamma'(t) = \gamma(t)u(t), \qquad \Lambda u'(t) + \mathrm{ad}_{u(t)}\Lambda u(t) = 0. \tag{2.14}$$

Similarly, the Jacobi equation and the index form can be written in terms of the Λ operator.

Lemma 2.8. *Let G be a quadratic Lie group with left-invariant metric induced by an operator Λ. Then in the notations of Proposition 2.4, the Jacobi equation* (2.7) *is written*

$$y'(t) + \mathrm{ad}_{u(t)}y(t) = z(t), \qquad \Lambda z'(t) + \mathrm{ad}_{u(t)}\Lambda z(t) + \mathrm{ad}_{z(t)}\Lambda u(t) = 0.$$

The index form is given by

$$I(y,y) = \int_0^\tau \langle \Lambda z(t) + \mathrm{ad}_{y(t)}\Lambda u(t), z(t)\rangle\, dt, \tag{2.15}$$

for variations $y(t)$ which vanish at $t = 0$ and $t = \tau$.

Proof. This is a straightforward consequence of Lemma 2.6 and Propositions 2.4 and 2.5, associated with the ad-invariance of $\langle\cdot,\cdot\rangle$. □

3 Conjugate points along nonsteady geodesics

3.1 General Lie groups

We first give the proof of Theorem 1.1, then discuss a simple consequence: a compact nonabelian Lie group with a left-invariant metric must have some positive sectional curvature.

Theorem 3.1. *Suppose $\gamma(t)$ is a solution of* (1.1)*, with velocity $u(t)$ nonconstant. If right multiplication by $\gamma(\tau)\gamma(0)^{-1}$ is an isometry of the left-invariant metric for some $\tau > 0$, then $\gamma(\tau)$ is conjugate to $\gamma(0)$.*

Proof of Theorem 1.1. By left-invariance, we may assume without loss of generality that $\gamma(0)$ is the identity. Let $u(t)$ be defined by the flow equation $\gamma'(t) = \gamma(t)u(t)$. By Proposition 2.3, $u(t)$ satisfies the angular momentum conservation law

$$u(t) = \mathrm{Ad}^\star_{\gamma(t)}u_0 \tag{3.1}$$

in terms of the initial condition $u(0) = u_0$. Differentiating the Euler equation (2.2) for $u(t)$ with respect to time gives

$$u''(t) - \mathrm{ad}^\star_{u(t)}u'(t) - \mathrm{ad}^\star_{u'(t)}u(t) = 0,$$

showing that $z_p(t) := u'(t)$ is a particular solution of the linearized Euler equation (2.7). Since $u'(t)$ is nowhere zero, this is a nontrivial solution.

From here we only need to find a solution $y(t)$ of the full system (2.7), satisfying $y(0) = y(\tau) = 0$. Since the equation in (2.7) for y is linear and nonhomogeneous, it can be solved using the standard technique of finding a particular solution and the general complementary solution. An obvious particular solution is $y_p(t) = u(t)$, since we have

$$\frac{d}{dt} y_p(t) + \mathrm{ad}_{u(t)} y_p(t) = u'(t) + 0 = z_p(t).$$

The general homogeneous solution may be found by rewriting (2.7) in the form

$$\frac{d}{dt}\Big(\mathrm{Ad}_{\gamma(t)} y(t)\Big) = \mathrm{Ad}_{\gamma(t)} z(t),$$

so we see that the general solution of the complementary homogeneous equation is $y_c(t) = \mathrm{Ad}_{\gamma(t)^{-1}} w_0$ for some vector $w_0 \in \mathfrak{g}$. Thus the general solution is

$$y(t) = y_p(t) + y_c(t) = u(t) + \mathrm{Ad}_{\gamma(t)^{-1}} w_0.$$

To have $y(0) = 0$ as desired, we choose $w_0 = -u_0$. Inserting (3.1) in this, we obtain a solution

$$y(t) = \mathrm{Ad}^\star_{\gamma(t)} u_0 - \mathrm{Ad}_{\gamma(t)^{-1}} u_0.$$

If right multiplication by $\gamma(\tau)$ is an isometry, then $\mathrm{Ad}^\star_{\gamma(\tau)} \mathrm{Ad}_{\gamma(\tau)} = I$, and we will obtain $y(\tau) = 0$. Clearly the corresponding Jacobi field $y(t)$ is nontrivial on $[0, \tau]$ since $z_p(t)$ is nontrivial. □

In order to state a Corollary of Theorem 1.1, recall that a Riemannian manifold M is said to have *dense closed geodesics* if for every $p \in M$ and every unit $v \in T_pM$ and every $\varepsilon > 0$, there is a $w \in T_pM$ such that $|v - w| < \varepsilon$ and the geodesic $\gamma(t) = \exp_p(tw)$ is closed (i.e., $\gamma(\tau) = p$ for some $\tau > 0$).

Examples of manifolds with dense closed geodesics include: compact manifolds with negative curvature, by the Anosov theorem; certain quotients of nilpotent Lie groups, all of which have curvatures of both signs (so long as they are not abelian) ([12], [15], [33], [36]); and of course many examples with positive curvature, such as $U(n)$ with the bi-invariant metric (see also [7]). By left-invariance, it is enough to check this condition when p is the identity.

Corollary 3.2. *If a (finite-dimensional) Lie group G with left-invariant metric has dense closed geodesics, then it must have positive curvature in some section at the identity, or it is abelian and flat.*

Proof. For every $v \in T_{\mathrm{id}}G$ there is a nearby vector w such that the geodesic in the direction of w is closed. If any such geodesic is nonsteady, the previous theorem gives a conjugate point along it, which implies there must be positive curvature somewhere along the geodesic. Otherwise every closed geodesic is steady.

If the geodesic $t \mapsto \exp_{\mathrm{id}}(tw)$ is steady, then we must have $\mathrm{ad}^\star_w w = 0$. Now the quadratic form $v \mapsto A(v) := \mathrm{ad}^\star_v v$ is continuous on a finite-dimensional Lie algebra, and density of closed geodesics implies that for every $v \in T_{\mathrm{id}}G$ and $\delta > 0$ there is a $w \in T_{\mathrm{id}}G$ such that $|v - w| < \delta$ and $A(w) = 0$. Hence we must have $A(v) = 0$ for all $v \in T_{\mathrm{id}}G$.

Now $\mathrm{ad}^\star_v v = 0$ for all $v \in T_{\mathrm{id}}G$ implies that the metric is bi-invariant, and thus the sectional curvature $K(u, v)$ is given by the well-known formula $K(u, v) = \frac{1}{4}|[u, v]|^2$. Thus either $[u, v] = 0$ for all $u, v \in T_{\mathrm{id}}G$, so that G is abelian and flat, or we again get some positive curvature. □

3.2 Quadratic Lie groups

In this section we prove Theorem 1.2. Suppose G is a quadratic Lie group, with metric g defined from an operator Λ and a bi-invariant bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ through (2.12). Assume $u(t)$ is a nonsteady solution of the Euler-Arnold equation (2.14).

We will construct a variation field y in the Lie algebra such that the induced vector field along the nonsteady geodesic defined by u makes the index form negative, thus proving the existence of conjugate points. This construction relies on conserved quantities which are direct analogues of the conservation laws for energy and angular momentum in the case of a rigid body, but exist on any quadratic Lie group.

Lemma 3.3. *Suppose g is generated by an operator Λ as in* (2.12)*. Then for any solution $u(t)$ of the Euler-Arnold equation* (2.14)*, the quantities*

$$k := \langle u(t), \Lambda u(t) \rangle \qquad \text{and} \qquad \ell := \langle \Lambda u(t), \Lambda u(t) \rangle \tag{3.2}$$

are constant in time.

Proof. A straightforward computation: by symmetry of Λ we have

$$\begin{aligned} k' &= 2\langle u, \Lambda u' \rangle = -2\langle u, \mathrm{ad}_u \Lambda u \rangle = 2\langle \mathrm{ad}_u u, \Lambda u \rangle = 0, \\ \ell' &= 2\langle \Lambda u, \Lambda u' \rangle = -2\langle \Lambda u, \mathrm{ad}_u \Lambda u \rangle = 2\langle \Lambda u, \mathrm{ad}_{\Lambda u} u \rangle = -2\langle \mathrm{ad}_{\Lambda u} \Lambda u, u \rangle = 0, \end{aligned}$$

using bi-invariance of $\langle \cdot, \cdot \rangle$ both times. □

Using these quantities, we can construct a family of orthogonal frames that include u, and such that the geodesic variations within the 2-planes orthogonal to u are good candidates to make the index form negative.

Lemma 3.4. *Suppose $u(t)$ is a nonsteady solution of the Euler-Arnold equation* (2.14)*. Let*

$$v_1(t) := u'(t)/g(u', u'), \qquad v_2(t) := k\Lambda u(t) - \ell u(t), \qquad v_3(t) := u(t). \tag{3.3}$$

Then v_1, v_2, v_3 are mutually orthogonal (though not generally orthonormal) in the metric g.

Proof. Since $u(t)$ is nonsteady, we know $u'(t)$ is never zero, so v_1 is well-defined. Since $k = g(u, u)$ is constant by Lemma 3.3, we have $g(u', u) = 0$, so v_1 and v_3 are orthogonal. Since $\ell = g(u, \Lambda u)$ is also constant, we find that u' is orthogonal to Λu. Hence v_1 is orthogonal to Λu, and therefore also to v_2. Finally v_2 is orthogonal to v_3 by definitions (3.2), since

$$g(v_2, v_3) = g(k\Lambda u - \ell u, u) = kg(\Lambda u, u) - \ell g(u, u) = k\ell - \ell k = 0.$$

□

We now want to compute the integrand of the index form (2.15) for a variational field y in the Lie algebra. We restrict to fields y that are linear combinations of $v_1(t)$ and $v_2(t)$, since including a term proportional to $v_3(t) = u(t)$ would only produce a larger index form as the reader can check. So we lose nothing by omitting it.

Lemma 3.5. *Let $\tau > 0$ and $y : [0,\tau] \to \mathfrak{g}$ be a test field with decomposition $y(t) = y_1(t)v_1(t) + y_2(t)v_2(t)$ with respect to the the fields v_1, v_2 defined by* (3.3)*. Define the function*

$$\zeta(t) := 1/\sqrt{g(u', u')} = \sqrt{g(v_1, v_1)}, \tag{3.4}$$

and the vector fields in $\mathfrak{g}$

$$z(t) := y'(t) + \mathrm{ad}_{u(t)}y(t), \quad w(t) := v_1'(t) + \mathrm{ad}_{u(t)}v_1(t), \quad x(t) := \Lambda w(t) + \mathrm{ad}_{v_1(t)}\Lambda u(t). \tag{3.5}$$

Then we have

$$z = \big(y_1' - \ell\zeta^{-2}y_2\big)v_1 + y_2'v_2 + y_1 w \tag{3.6}$$

$$\Lambda z + \mathrm{ad}_y\Lambda u = y_1'\Lambda v_1 + y_2'\Lambda v_2 + y_1 x \tag{3.7}$$

Proof. The formula for z is given by

$$\begin{aligned} z &= y_1'v_1 + y_2'v_2 + y_1(v_1' + \mathrm{ad}_u v_1) + y_2(v_2' + \mathrm{ad}_u v_2) \\ &= y_1'v_1 + y_2'v_2 + y_1 w + y_2(v_2' + \mathrm{ad}_u v_2) \end{aligned}$$

So to prove formula (3.6), we just need to simplify this last term. We have using definitions (3.3)

$$\begin{aligned} v_2' + \mathrm{ad}_u v_2 &= k\Lambda u' - \ell u' + \mathrm{ad}_u(k\Lambda u - \ell u) \\ &= -k\mathrm{ad}_u\Lambda u - \ell u' + k\mathrm{ad}_u\Lambda u \\ &= -\ell\zeta^{-2}v_1. \end{aligned}$$

Plugging this in gives (3.6). Similarly to prove the formula (3.7), we compute

$$\begin{aligned} \Lambda z + \mathrm{ad}_y\Lambda u &= y_1'\Lambda v_1 + y_2'\Lambda v_2 + y_1\Lambda w + y_2\Lambda(v_2' + \mathrm{ad}_u v_2) + y_1\mathrm{ad}_{v_1}\Lambda u + y_2\mathrm{ad}_{v_2}\Lambda u \\ &= y_1'\Lambda v_1 + y_2'\Lambda v_2 + y_1 x + y_2(-\ell\Lambda u' + \mathrm{ad}_{v_2}\Lambda u). \end{aligned}$$

We will be done once we show that the term attached to y_2 in the last line is zero. To do this, recall that we have $v_2 = k\Lambda u - \ell u$, so that

$$-\ell\Lambda u' + \mathrm{ad}_{v_2}\Lambda u = \ell\mathrm{ad}_u\Lambda u + k\mathrm{ad}_{\Lambda u}\Lambda u - \ell\mathrm{ad}_u\Lambda u = 0.$$

using the Euler-Arnold equation (2.14). □

Lemma 3.6. *The vector fields $w(t)$ and $x(t)$ defined in* (3.5) *satisfy the following identities:*

$$\begin{aligned} &\langle w, \Lambda v_1\rangle = 0, \qquad &&\langle w, \Lambda v_2\rangle = k\langle \Lambda w, \Lambda u\rangle, \qquad &&\langle w, \Lambda v_3\rangle = 0 \\ &\langle x, v_1\rangle = 0, &&\langle x, v_2\rangle = k\langle \Lambda w, \Lambda u\rangle - \ell, &&\langle x, v_3\rangle = 1 \end{aligned}$$

Proof. We start with the inner products with respect to $v_3 = u$. Recall that $w = v_1' + \mathrm{ad}_u v_1$ with $v_1 = \zeta^2 u'$, for $\zeta^2 = g(u', u')^{-1} = \langle u', \Lambda u'\rangle^{-1}$. Thus we can write

$$w = \zeta^2\big(u'' + 2\tfrac{\zeta'}{\zeta}u' + \mathrm{ad}_u u'\big). \tag{3.8}$$

Hence we get

$$\begin{aligned}\langle w, \Lambda v_3\rangle &= \zeta^2\langle u'' + 2\tfrac{\zeta'}{\zeta}u' + \mathrm{ad}_u u', \Lambda u\rangle = \zeta^2\big(\langle \Lambda u'', u\rangle - \langle u', \mathrm{ad}_u \Lambda u\rangle\big)\\ &= \zeta^2\big(\tfrac{d}{dt}\langle \Lambda u', u\rangle - \langle u', \Lambda u'\rangle + \langle u', \Lambda u'\rangle\big) = 0,\end{aligned}$$

using the conservation of $k = \langle u, \Lambda u\rangle$ and the Euler-Arnold equation (2.14), together with ad-invariance of $\langle \cdot, \cdot\rangle$. Similarly,

$$\begin{aligned}\langle x, v_3\rangle &= \langle \Lambda w + \mathrm{ad}_{v_1}\Lambda u, u\rangle = \zeta^2\langle \mathrm{ad}_{u'}\Lambda u, u\rangle\\ &= -\zeta^2\langle \mathrm{ad}_u \Lambda u, u'\rangle = \zeta^2\langle \Lambda u', u'\rangle = 1.\end{aligned}$$

For the inner product with v_1, we use (3.8) and the derivative of the Euler-Arnold equation (2.14) to get

$$\begin{aligned}\langle w, \Lambda v_1\rangle &= \zeta^4\langle u'' + 2\tfrac{\zeta'}{\zeta}u' + \mathrm{ad}_u u', \Lambda u'\rangle\\ &= \zeta^4\big(- \langle u', \Lambda u''\rangle + \langle \mathrm{ad}_u u', \Lambda u'\rangle\big) = \zeta^4\big(\langle u', \mathrm{ad}_{u'}\Lambda u + \mathrm{ad}_u \Lambda u'\rangle + \langle \mathrm{ad}_u u', \Lambda u'\rangle\big)\\ &= \zeta^4\big(- \langle \mathrm{ad}_{u'}u', \Lambda u\rangle + \langle -\mathrm{ad}_u u' + \mathrm{ad}_u u', \Lambda u'\rangle\big) = 0,\end{aligned}$$

where we used the fact that

$$\zeta'/\zeta^3 = -\langle u'', \Lambda u'\rangle.$$

This gives

$$\langle x, v_1\rangle = \langle \Lambda w, v_1\rangle + \langle \mathrm{ad}_{v_1}\Lambda u, v_1\rangle = \langle w, \Lambda v_1\rangle - \langle \Lambda u, \mathrm{ad}_{v_1} v_1\rangle = 0.$$

Finally for coordinates along v_2 there is not much to do: we recall $v_2 = k\Lambda u - \ell v_3$, so that

$$\langle w, \Lambda v_2\rangle = k\langle \Lambda w, \Lambda u\rangle - \ell\langle w, \Lambda v_3\rangle = k\langle \Lambda w, \Lambda u\rangle,$$

using our previous computation. Similarly we get

$$\begin{aligned}\langle x, v_2\rangle - k\langle \Lambda w, \Lambda u\rangle &= \langle \mathrm{ad}_{v_1}\Lambda u, v_2\rangle = -\zeta^2\langle \mathrm{ad}_{\Lambda u}u', k\Lambda u - \ell u\rangle\\ &= \zeta^2\langle u', \mathrm{ad}_{\Lambda u}(k\Lambda u - \ell u)\rangle = \zeta^2\langle u', -\ell\mathrm{ad}_{\Lambda u}u\rangle\\ &= \ell\zeta^2\langle u', \mathrm{ad}_u\Lambda u\rangle = -\ell\zeta^2\langle u', \Lambda u'\rangle = -\ell.\end{aligned}$$

□

We now combine the last several lemmas to simplify the index form along variations spanned by v_1 and v_2.

Proposition 3.7. *Let $y(t) = y_1(t)v_1(t) + y_2(t)v_2(t)$ be a variation as described in Lemma 3.5, vanishing at $t = 0$ and $t = \tau$, and let $\zeta(t)$ be the function defined by (3.4), and*

$$\alpha(t) = k\langle \Lambda u(t), \Lambda w(t)\rangle, \qquad \beta(t) = g(v_2(t), v_2(t)).$$

Then, the index form $I(y, y)$ from Lemma 2.8 becomes

$$I(y, y) = \int_0^\tau \zeta^2(y_1')^2 + \beta\left(y_2' + \frac{\alpha}{\beta}y_1\right)^2 + y_1^2\left(\langle w, x\rangle - \frac{\alpha^2}{\beta}\right)dt. \tag{3.9}$$

Proof. By equations (3.6) and (3.7), the integrand of the index form (2.15) is given by

$$\begin{aligned}\langle z, \Lambda z + \mathrm{ad}_y \Lambda u\rangle &= \big\langle (y_1' - \ell\zeta^{-2}y_2)v_1 + y_2'v_2 + y_1 w,\; y_1'\Lambda v_1 + y_2'\Lambda v_2 + y_1 x\big\rangle \\ &= (y_1' - \ell\zeta^{-2}y_2)y_1'\langle v_1, \Lambda v_1\rangle + (y_2')^2\langle v_2, \Lambda v_2\rangle \\ &+ y_1\langle (y_1' - \ell\zeta^{-2}y_2)v_1 + y_2'v_2, x\rangle + y_1\langle w, y_1'\Lambda v_1 + y_2'\Lambda v_2\rangle + y_1^2\langle w, x\rangle.\end{aligned}$$

Now we apply Lemma 3.6 to simplify these terms. We obtain

$$\begin{aligned}\langle z, \Lambda z + \mathrm{ad}_y \Lambda u\rangle &= (y_1' - \ell\zeta^{-2}y_2)y_1' g(v_1, v_1) + g(v_2, v_2)(y_2')^2 \\ &\quad + y_1 y_2'\big(\langle v_2, x\rangle + \langle w, \Lambda v_2\rangle\big) + y_1^2\langle w, x\rangle \\ &= \zeta^2(y_1')^2 + \beta(y_2')^2 + 2k\langle \Lambda w, \Lambda u\rangle y_1 y_2' + \langle w, x\rangle y_1^2 - \ell(y_2 y_1' + y_1 y_2').\end{aligned}$$

The last term attached to ℓ will integrate to zero on $[0, \tau]$ since it is a total time derivative of $y_1 y_2$, which vanishes at $t = 0$ and $t = \tau$. Now completing the square in what remains,

$$\begin{aligned}\langle z, \Lambda z + \mathrm{ad}_y \Lambda u\rangle &= \zeta^2(y_1')^2 + \beta\Big(y_2' + k\beta^{-1}\langle \Lambda w, \Lambda u\rangle y_1\Big)^2 \\ &\quad + \Big(\langle w, x\rangle - k^2\beta^{-1}\langle \Lambda w, \Lambda u\rangle^2\Big)y_1^2,\end{aligned}$$

plus a term that integrates to zero, and so the integral on $[0, \tau]$ is given by (3.9). □

We now prove a lemma which will be used for establishing negativity of the index form (3.9) for some choice of vector field $y(t) = y_1(t)v_1(t) + y_2(t)v_2(t)$. The auxiliary condition that the integral of $f\xi$ is zero is needed in order to guarantee that the second term of the integrand on the index form (3.9) can be made to vanish in order to reduce the criterion to one involving a single function of time $y_1(t)$.

Lemma 3.8. *Suppose ζ and ϕ are real functions on $[0, \infty)$, with ζ positive and such that one of the two following conditions is satisfied:*

(i) the functions ζ and ϕ are bounded respectively above and below by positive constants,

(ii) the following quantity is bounded below by a positive constant for all $t \geq 0$:

$$\psi(t) := \frac{\phi(t)}{\zeta(t)^2} - \frac{\zeta''(t)}{\zeta(t)}$$

Then for any function $\xi(t)$, and for sufficiently large $\tau > 0$, there is a function $f(t)$ with $f(0) = f(\tau) = 0$, together with $\int_0^\tau f(t)\xi(t)\,dt = 0$, and such that

$$I := \int_0^\tau \Big(\zeta(t)^2 f'(t)^2 - \phi(t)f(t)^2\Big)\,dt < 0.$$

Proof. Let $\xi(t)$ be a function. If the first condition is fulfilled, i.e., $\zeta(t)^2 \leq a^2$ and $\phi(t) \geq b^2$ for some real numbers a, b, then

$$I \leq \int_0^\tau a^2 f'(t)^2 - b^2 f(t)^2\,dt.$$

Now choose $f(t) = k_1 \sin\frac{\pi t}{\tau} + k_2 \sin\frac{2\pi t}{\tau}$. Clearly $f(0) = f(\tau) = 0$. We may obviously choose a nontrivial combination k_1 and k_2 such that $\int_0^\tau f(t)\xi(t)\,dt = 0$. The integral I is straightforward to compute, and we get

$$I = \frac{a^2\pi^2}{\tau}\,(k_1^2 + 4k_2^2) - \frac{b^2\tau}{2}\,(k_1^2 + k_2^2),$$

which will be negative for sufficiently large τ.

If the second condition is fulfilled, we write $f(t) = h(t)/\zeta(t)$ for convenience and get

$$\begin{aligned} I &= \int_0^\tau \zeta(t)^2 \left(\frac{h'(t)}{\zeta(t)} - \frac{h(t)\zeta'(t)}{\zeta(t)^2}\right)^2 - \frac{\phi(t)}{\zeta(t)^2}\,h(t)^2\,dt \\ &= \int_0^\tau h'(t)^2 - \left(\frac{\phi(t)}{\zeta(t)^2} - \frac{\zeta''(t)}{\zeta(t)}\right) h(t)^2 - 2h(t)h'(t)\,\frac{\zeta'(t)}{\zeta(t)} + h(t)^2 \left(\frac{\zeta'(t)^2}{\zeta(t)^2} - \frac{\zeta''(t)}{\zeta(t)}\right) dt \\ &= \int_0^\tau h'(t)^2 - \psi(t)h(t)^2 - \frac{d}{dt}\left(h(t)^2\,\frac{\zeta'(t)}{\zeta(t)}\right) dt \\ &= \int_0^\tau h'(t)^2 - \psi(t)h(t)^2\,dt, \end{aligned}$$

where the total derivative cancels since h vanishes at the endpoints. By assumption $\psi(t) \geq c^2$ for some real number c, so that

$$I \leq \int_0^\tau h'(t)^2 - c^2 h(t)^2\,dt,$$

and we can choose for h the same function as in the first case, with k_1 and k_2 such that $\int_0^\tau h(t)\xi(t)/\zeta(t)\,dt = 0$, making once again I negative for sufficiently large τ. □

We can now finally get our criterion for existence of conjugate points.

Theorem 3.9. *Let G be a quadratic Lie group with a left-invariant metric g given by* (2.12) *for an operator Λ with a bi-invariant nondegenerate bilinear form $\langle\cdot,\cdot\rangle$. Let $u(t)$ be a non-steady solution of the Euler equation, and let*

$$k := g(u, u), \qquad \ell := g(u, \Lambda u),$$

which are conserved quantities. Define the following vector fields on the Lie algebra

$$v_1 := u'/g(u', u'), \quad v_2 := k\Lambda u - \ell u, \quad w := v_1' + \mathrm{ad}_u v_1, \quad x := \Lambda w + \mathrm{ad}_{v_1}\Lambda u.$$

and the following functions $[0, \infty) \to \mathbb{R}$,

$$\zeta(t) := \sqrt{g(v_1, v_1)}, \qquad \phi(t) := \frac{k^2\langle \Lambda w, \Lambda u\rangle^2}{g(v_2, v_2)} - \langle w, x\rangle, \qquad \psi(t) := \frac{\phi(t)}{\zeta(t)^2} - \frac{\zeta''(t)}{\zeta(t)}.$$

Assume that one of the two following conditions is satisfied: (i) the functions ζ and ϕ are bounded respectively above and below by positive constants, or (ii) ψ is bounded below by a positive number. Then there is eventually a conjugate point along the geodesic $\gamma(t)$ corresponding to $u(t)$ via (1.1). *In particular, γ is not globally minimizing.*

Proof. Let $\alpha(t)$, $\beta(t)$ be as defined in Proposition 3.7 and $\xi(t) := \frac{\alpha(t)^2}{\beta(t)}$. Then by Lemma 3.8, there exists a function f and a $\tau > 0$ such that

$$I := \int_0^\tau \zeta(t)^2 f'(t)^2 - \phi(t) f(t)^2 \, dt < 0, \tag{3.10}$$

together with $f(0) = f(\tau) = 0$ and $\int_0^\tau \xi(t) f(t) dt = 0$. Setting

$$y_1(t) = f(t) \qquad \text{and} \qquad y_2'(t) = -\frac{\alpha(t)^2}{\beta(t)} f(t),$$

we obtain a field $y = y_1 v_1 + y_2 v_2$ that vanishes at $t = 0$ and $t = \tau$, and for which the index form (3.9) is precisely (3.10). □

We will give examples of applications in Sections 5 and 6. In particular in any Berger-Cheeger group the quantities $\zeta(t)$ and $\phi(t)$ are constant. On $SO(n)$, condition (i) is satisfied along most geodesics, except those joined to unstable solutions where condition (ii) is required; we will present an explicit example for $n = 3$.

4 Conjugate points along steady geodesics

4.1 General Lie groups

Here we prove Theorem 4.1, restated below.

Theorem 4.1. *Suppose u_0 is a steady solution of the Euler-Arnold equation* (1.1)*, i.e., that $\mathrm{ad}^\star_{u_0} u_0 = 0$. Let L and F be the linear operators on the Lie algebra $\mathfrak{g}$ defined by*

$$L(v) := \mathrm{ad}_{u_0} v, \qquad F(v) = \mathrm{ad}^\star_{u_0} v + \mathrm{ad}^\star_v u_0.$$

Then both operators map the g-orthogonal complement of u_0 to itself. If there is an operator R defined on this orthogonal complement such that $RF + LR = I$, then there is a conjugate point along the geodesic if and only if for some $\tau > 0$ we have

$$\det\left(e^{\tau L} R e^{\tau F} - e^{-\tau L} R e^{-\tau F}\right) = 0.$$

Proof. Let $u_0 \in \mathfrak{g}$ be a steady solution of the Euler-Arnold equation (1.1), i.e. such that $\mathrm{ad}^\star_{u_0} u_0 = 0$. There are conjugate points along the corresponding geodesic γ if there is a time $\tau > 0$ and vector fields $y(t)$ and $z(t)$ of the Lie algebra satisfying $y(0) = y(\tau) = 0$ and the system of equations (2.6), that is,

$$y' + L(y) = z, \qquad z' = F(z) \tag{4.1}$$

in terms of the linear operators on the Lie algebra $\mathfrak{g}$

$$L(y) = \mathrm{ad}_{u_0} y, \qquad F(y) = \mathrm{ad}^\star_{u_0} y + \mathrm{ad}^\star_y u_0. \tag{4.2}$$

Both these operators map into the g-orthogonal complement of u_0. Indeed, for any $y \in \mathfrak{g}$,

$$g(L(y), u_0) = g(y, \mathrm{ad}^\star_{u_0} u_0) = 0$$
$$g(F(y), u_0) = g(y, \mathrm{ad}_{u_0} u_0) + g(u_0, \mathrm{ad}_y u_0) = -g(\mathrm{ad}^\star_{u_0} u_0, y) = 0.$$

Now, if there is an operator R defined on this orthogonal complement such that $RF + LR = I$, then every solution of (4.1) with $y(0), z(0) \in u_0^\perp$ is obtained as

$$y(t) = Re^{tF}x_0 + e^{-tL}w_0,$$

for some choice of $w_0, x_0 \in u_0^\perp$, since the first part solves the nonhomogeneous equation for y, while the second part solves the homogeneous equation.

A conjugate point occurs if and only if there is a time τ such that $y(0) = y(\tau) = 0$, or equivalently, $y(-\tau) = y(\tau)$, since the geodesic is steady and this time translation has no impact. This means

$$Re^{\tau F}x_0 + e^{-\tau L}w_0 = 0$$
$$Re^{-\tau F}x_0 + e^{\tau L}w_0 = 0$$

Left multiplying by $e^{\tau L}$ and $e^{-\tau L}$ respectively the first and second equation, and eliminating w_0, we obtain that the condition for the existence of a conjugate point along the steady geodesic γ is the existence of a nontrivial $x_0 \in \mathfrak{g}$ such that

$$(e^{\tau L}Re^{\tau F} - e^{-\tau L}Re^{-\tau F})x_0 = 0,$$

which is equivalent to asking that the determinant of the matrix multiplying x_0 be zero, as claimed. □

In practice we will see that this form is especially convenient when L and F are block-diagonal matrices, so that the matrix exponentials can be computed easily.

4.2 Quadratic Lie groups

In this section, we prove Theorem 4.2, which gives a criterion for conjugate points along the simplest steady geodesics on quadratic Lie groups, those generated by eigenvectors of the operator Λ.

Theorem 4.2. *Suppose G is a quadratic group, with a left-invariant metric g given by* (2.12) *for an operator Λ with a bi-invariant nondegenerate bilinear form $\langle \cdot, \cdot \rangle$. Let u_0 be an eigenvector of Λ with $\Lambda u_0 = \lambda u_0$, and let $L = \mathrm{ad}_{u_0}$. If L^2 commutes with Λ, then there is eventually a conjugate point along the geodesic $\gamma(t)$ solving the geodesic equation* (2.14)*.*

Proof. By assumption, both L^2 and Λ can be diagonalized in the same basis $\{w_1, \ldots, w_n\}$, orthonormal with respect to the bi-invariant form. Since L is antisymmetric in the bi-invariant form, we know L^2 is symmetric and nonpositive. Note that if w is an eigenvector of L^2 with $L^2(w) = -\epsilon^2 w$, then $L(w)$ is an eigenvector with the same eigenvalue, since $L^2(L(w)) = L(L^2(w)) = -\epsilon^2 L(w)$; and Lw is orthogonal to w: $\langle w, Lw \rangle = -\langle Lw, w \rangle$ since L is antisymmetric, thus $\langle w, Lw \rangle = 0$. Therefore L has eigenspaces of even dimensions, and denoting by m the number of nonzero eigenvalues of L^2, we can order the eigenvectors $\{w_1, \ldots, w_{2m}, w_{2m+1}, \ldots, w_n\}$ so that

$$L(w_{2j-1}) = \epsilon_j w_{2j}, \qquad L(w_{2j}) = -\epsilon_j w_{2j-1},$$
$$\Lambda(w_{2j-1}) = \alpha_j w_{2j-1}, \qquad \Lambda(w_{2j}) = \beta_j w_{2j}, \qquad 1 \le j \le m,$$

and

$$L(w_k) = 0 \qquad \Lambda(w_k) = \gamma_k w_k, \qquad 2m+1 \le k \le n,$$

with the vector u_0 itself a constant multiple of one of the w_k, and all numbers α, β, γ, and ϵ strictly positive. See for example Greub [20], Section 8.4.

Recall from Theorem 4.1 that conjugate points occur along the steady geodesic defined by u_0 if and only if the determinant

$$\det\left(e^{tL} R e^{tF} - e^{-tL} R e^{-tF}\right)$$

is zero for some $t = \tau > 0$, with the operators defined by (4.2). Here we have $L(z) = \mathrm{ad}_{u_0} z$ and

$$\begin{aligned} F(z) &= \mathrm{ad}^\star_{u_0}(z) + \mathrm{ad}^\star_z u_0 = -\Lambda^{-1}(\mathrm{ad}_{u_0}\Lambda z + \mathrm{ad}_z \Lambda u_0) \\ &= -\Lambda^{-1}(\mathrm{ad}_{u_0}\Lambda z + \lambda \mathrm{ad}_z u_0) \\ &= -\Lambda^{-1} L(\Lambda - \lambda I)(z), \end{aligned}$$

where we have used (2.13). F is given in the orthonormal basis $\{w_1, \ldots, w_n\}$ by

$$F(w_{2j-1}) = -\frac{\epsilon_j(\alpha_j - \lambda)}{\beta_j} w_{2j}, \qquad F(w_{2j}) = \frac{\epsilon_j(\beta_j - \lambda)}{\alpha_j} w_{2j-1}, \qquad 1 \le j \le m,$$
$$F(w_k) = 0, \qquad 2m+1 \le k \le n.$$

Now let $\mathfrak{h}$ denote the span of $\{w_1, \ldots, w_{2m}\}$, and observe that L, Λ, and F all map $\mathfrak{h}$ to itself. We thus restrict F and L to $\mathfrak{h}$ and note that L is invertible on this subspace, so that

$$R = \lambda^{-1} L^{-1} \Lambda \quad \Longrightarrow \quad RF + LR = I.$$

In the basis, R is given by

$$R(w_{2j-1}) = -\frac{\alpha_j}{\lambda \epsilon_j} w_{2j}, \qquad R(w_{2j}) = \frac{\beta_j}{\lambda \epsilon_j} w_{2j-1}.$$

We thus find that the matrices L, R, and F restricted to $\mathfrak{h}$ can be written as $(2m) \times (2m)$ matrices consisting of 2×2 nonzero blocks along the diagonal, and zero everywhere else. Hence the matrices e^{tL} and e^{tF} also split this way, and the determinant from Theorem 4.1 splits into a product:

$$\det\left(e^{tL} R e^{tF} - e^{-tL} R e^{-tF}\right) = \prod_{j=1}^{k} \det\left(e^{tL_j} R_j e^{tF_j} - e^{-tL_j} R_j e^{-tF_j}\right),$$

where each 2×2 matrix acting in $\mathfrak{h}_j = \operatorname{span}\{w_{2j-1}, w_{2j}\}$ is given by

$$L_j = \begin{pmatrix} 0 & -\epsilon_j \\ \epsilon_j & 0 \end{pmatrix}, \qquad F_j = \begin{pmatrix} 0 & \frac{\epsilon_j(\beta_j-\lambda)}{\alpha_j} \\ -\frac{\epsilon_j(\alpha_j-\lambda)}{\beta_j} & 0 \end{pmatrix}, \qquad R_j = \begin{pmatrix} 0 & \frac{\beta_j}{\lambda\epsilon_j} \\ -\frac{\alpha_j}{\lambda\epsilon_j} & 0 \end{pmatrix}.$$

Obviously the product of determinants is zero if and only if at least one of them is zero; thus it is sufficient to fix a $j \in \{1, \ldots, m\}$. The matrix exponential of tL_j is easy: since L_j is always antisymmetric, we have

$$e^{tL_j} = \cos(\epsilon_j t)I + \epsilon_j^{-1}\sin(\epsilon_j t)L_j.$$

On the other hand the form of the matrix exponential of τF_j depends on the quantity

$$d_j = \det F_j = \frac{\epsilon_j^2(\beta_j-\lambda)(\alpha_j-\lambda)}{\alpha_j\beta_j}.$$

We have $e^{tF_j} = c_j(t)I + s_j(t)F$, where the pair of generalized trigonometric functions are

$$\big(c_j(t), s_j(t)\big) := \begin{cases} (\cosh rt, r^{-1}\sinh rt) & \text{if } d_j = -r^2, \\ (1, t) & \text{if } d_j = 0, \\ (\cos rt, r^{-1}\sin rt) & \text{if } d_j = r^2. \end{cases}$$

It is then easy to see that each matrix $e^{tL_j}R_je^{tF_j} - e^{-tL_j}R_je^{-tF_j}$ is *diagonal*, and so the condition in Theorem 4.1 becomes simply verifying that at least one of the following quantities is zero at some $t = \tau$:

$$f_j(t) = \sin(\epsilon_j t)c_j(t) - \frac{\epsilon_j(\alpha_j-\lambda)}{\alpha_j}\, s_j(t)\cos(\epsilon_j t)$$
$$g_j(t) = \sin(\epsilon_j t)c_j(t) - \frac{\epsilon_j(\beta_j-\lambda)}{\beta_j}\, s_j(t)\cos(\epsilon_j t).$$

Note that for small positive values of t, both these functions are small and positive.

If $d_j = r^2 > 0$, then $(\alpha_j - \lambda)$ and $(\beta_j - \lambda)$ have the same sign. The functions become

$$f_j(t) = \sin(\epsilon_j t)\cos(rt) - \operatorname{sign}(\alpha_j-\lambda)\sqrt{\frac{\beta_j|\alpha_j-\lambda|}{\alpha_j|\beta_j-\lambda|}}\,\sin(rt)\cos(\epsilon_j t)$$
$$g_j(t) = \sin(\epsilon_j t)\cos(rt) - \operatorname{sign}(\beta_j-\lambda)\sqrt{\frac{\alpha_j|\beta_j-\lambda|}{\beta_j|\alpha_j-\lambda|}}\,\sin(rt)\cos(\epsilon_j t).$$

We can simplify the following positive linear combination of $f(t)$ and $g(t)$ to get

$$\sqrt{\alpha_j|\beta_j-\lambda|}\, f_j(t) + \sqrt{\beta_j|\alpha_j-\lambda|}\, g_j(t) = \left(\sqrt{\alpha_j|\beta_j-\lambda|} + \sqrt{\beta_j|\alpha_j-\lambda|}\right)\sin(\epsilon_j \mp r)t.$$

Here the $\mp$ sign is negative if both $(\alpha_j - \lambda)$ and $(\beta_j - \lambda)$ are positive, and vice versa. Since we can obviously make this negative for some choice $t = \tau$, we find that at least one of $f_j(t)$

or $g_j(t)$ must have been negative at this time. Hence at least one of them must have crossed the axis already.

The situation when $d_j = -r^2$ is simpler, when $(\alpha_j - \lambda)$ and $(\beta_j - \lambda)$ have opposite signs. In this case the functions become

$$f_j(t) = \sin(\epsilon_j t)\cosh(rt) - \operatorname{sign}(\alpha_j - \lambda)\sqrt{\frac{\beta_j|\alpha_j - \lambda|}{\alpha_j|\beta_j - \lambda|}}\sinh(rt)\cos(\epsilon_j t)$$

$$g_j(t) = \sin(\epsilon_j t)\cosh(rt) - \operatorname{sign}(\beta_j - \lambda)\sqrt{\frac{\alpha_j|\beta_j - \lambda|}{\beta_j|\alpha_j - \lambda|}}\sinh(rt)\cos(\epsilon_j t),$$

and we simply plug in $t = \tau = \pi/\epsilon_j$ and multiply to get

$$f_j\left(\frac{\pi}{\epsilon_j}\right) g_j\left(\frac{\pi}{\epsilon_j}\right) = -\sinh^2\left(\frac{\pi r}{\epsilon_j}\right) < 0,$$

and thus either $f(\tau)$ or $g(\tau)$ must be negative, so it must have crossed the axis earlier. The case where $\alpha_j = \lambda$ or $\beta_j = \lambda$ is trivial. □

In the context of Theorem 4.2 above, we make the following remark on the Misiołek criterion (2.11), which explains why it captures some conjugate points but not others, and its connection with stability. Note that we have stability of U_0 in the Eulerian sense if and only if all eigenvalues of F are nonpositive, corresponding to $d_j \le 0$ for all j. This happens if for example λ is smaller than all α_j, β_j or larger than all of them.

Remark 4.3. *If u_0 is a steady solution of the Euler equation on a quadratic Lie group, corresponding to eigenvalue λ of Λ, the Misiołek criterion is written in terms of $L = \operatorname{ad}_{u_0}$ as*

$$0 > g(\operatorname{ad}_v u_0 + \operatorname{ad}_v^\star u_0, \operatorname{ad}_v u_0) = \langle \Lambda \operatorname{ad}_v u_0 - \lambda \operatorname{ad}_v u_0, \operatorname{ad}_v u_0\rangle = \langle (\Lambda - \lambda I)Lv, Lv\rangle.$$

Thus, we see that for steady solutions corresponding to, e.g., the smallest eigenvalue λ of Λ, the Misiołek criterion fails to detect the existence of conjugate points, that do exist when L^2 commutes with Λ, as for the generalized rigid body metric studied in the following section.

5 Generalized rigid body metric on rotations

The motion of a three-dimensional free rigid body has a long history, going back to Euler himself. Ignoring translations, at any point in time the body is rotating around an axis, but the axis itself can vary with time, which leads to fairly complicated phenomena. Already in three dimensions, the general equations of motion cannot be integrated by elementary functions, but require the so-called Jacobi elliptic functions [29].

This system admits a natural generalization to n dimensions which was first written down by Frahm [19], and has since been studied from many points of view. It is known to be completely integrable as it possesses a bihamiltonian structure (see Manakov [35], Mishchenko & Fomenko [18], Ratiu [41]). Stability of its stationary solutions has also been studied by many authors (see, e.g., [4], [17], [27], [28]), and remains an open problem when viewed in full generality.

In this section, we begin by showing that the Ricci curvature of the kinetic energy metric describing the n-dimensional rigid body system on $SO(n)$ is everywhere positive, leading to the existence of conjugate points along any geodesic, steady or nonsteady, by a result of Gromoll and Meyer [21]. This motivates the study of a larger class of metrics on $SO(n)$ that generalize the classical rigid body metrics in a natural way, while admitting some negative Ricci curvatures. We will see that our criterion finds conjugate points along steady geodesics even in the presence of negative curvature. The main takeaway here is that our criterion works through a different mechanism than existing methods, since it detects conjugate points not by an averaging method (cf. Gromoll & Meyer [21]), and is also not tied to a specific curvature formula or spectral condition (see Remark 4.3).

Consider the group of rotations $SO(n)$. A basis of its Lie algebra $\mathfrak{so}(n)$ is given by the matrices $\{e_{ij}\}_{1\leq i<j\leq n}$, where e_{ij} is the matrix full of zeros except for -1 in position (i,j) and 1 in position (j,i). We consider a symmetric operator $\Lambda : \mathfrak{so}(n) \to \mathfrak{so}(n)$ with eigenvectors e_{ij}, i.e., such that there exist real numbers λ_{ij} with

$$\Lambda e_{ij} = \lambda_{ij} e_{ij}, \qquad 1 \leq i < j \leq n.$$

For convenience in the curvature formula later, we define $\lambda_{ji} = \lambda_{ij}$. We equip $SO(n)$ with the left-invariant Riemannian metric generated on the Lie Algebra by

$$g(u,v) := \langle u, \Lambda v\rangle, \quad \text{where} \quad \langle u,v\rangle = \tfrac{1}{2}\mathrm{Tr}(uv^\top), \qquad u,v \in \mathfrak{so}(n), \tag{5.1}$$

is the bi-invariant metric and Tr is the trace. We refer to this metric as the generalized rigid body metric, since when $\Lambda(u) := \frac{1}{2}(Mu + uM)$ with M a symmetric matrix with positive eigenvalues $\mu_1, \ldots, \mu_n$, it is the kinetic energy metric, describing the rotations of a rigid body with principal moments of inertia μ_i. We will consider this special case at the end of this section.

5.1 Ricci curvature

We start by computing the Ricci curvature. We first need the following lemma.

Lemma 5.1. *The Lie brackets between the eigenvectors $\{e_{ij}\}_{1\leq i<j\leq n}$ are given by*

$$\begin{aligned} &\forall i<j<k, \quad [e_{ij}, e_{ik}] = e_{jk}, \quad [e_{ik}, e_{jk}] = e_{ij}, \quad [e_{jk}, e_{ij}] = e_{ik}, \\ &\textit{if} \quad \{i,j\} \cap \{k,\ell\} = \emptyset, \quad \textit{then} \quad [e_{ij}, e_{k\ell}] = 0. \end{aligned} \tag{5.2}$$

Proof. For any i,j,k,ℓ and any p,q, we have that the (p,q) component of e_{ij} is given by $(e_{ij})_{pq} = -\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}$, and a straightforward computation gives

$$[e_{ij}, e_{k\ell}]_{pq} = \sum_m (e_{ij})_{pm}(e_{k\ell})_{mq} - \sum_n (e_{k\ell})_{pn}(e_{ij})_{nq} = (\delta_{ik}e_{j\ell} - \delta_{jk}e_{i\ell} + \delta_{j\ell}e_{ik} + \delta_{i\ell}e_{kj})_{pq}.$$

□

Now we can compute the Ricci curvature.

Proposition 5.2. *The Ricci curvature tensor of the generalized rigid body metric* (5.1) *on* $SO(n)$ *is diagonal in the basis* $\{e_{ij}\}_{1\le i<j\le n}$ *with diagonal terms*

$$\mathrm{Ric}(e_{ij}, e_{ij}) = \sum_{k\neq i,j} \frac{(\lambda_{ij} - \lambda_{ik} + \lambda_{jk})(\lambda_{ij} + \lambda_{ik} - \lambda_{jk})}{2\lambda_{ik}\lambda_{jk}}$$

Remark 5.3. *In particular, for the standard case where the metric corresponds to the kinetic energy of a rigid body with moments of inertia* $\mu_1, \dots \mu_n > 0$*, then* $\lambda_{ij} = \frac{\mu_i+\mu_j}{2}$ *and we obtain*

$$\mathrm{Ric}(e_{ij}, e_{ij}) = \sum_{k\neq i,j} \frac{2\mu_i\mu_j}{(\mu_i + \mu_k)(\mu_j + \mu_k)}.$$

Thus the Ricci curvature of the rigid body metric on $SO(n)$ *is everywhere positive.*

Proof. Using Lemmas 2.6 and 5.1, we see that the operator $\mathrm{ad}^\star$ takes the following values:

$$\begin{aligned} \forall i<j<k, \quad \mathrm{ad}^\star_{e_{ik}} e_{ij} &= \frac{\lambda_{ij}}{\lambda_{jk}} e_{jk}, \quad \mathrm{ad}^\star_{e_{jk}} e_{ik} = \frac{\lambda_{ik}}{\lambda_{ij}} e_{ij}, \quad \mathrm{ad}^\star_{e_{ij}} e_{jk} = \frac{\lambda_{jk}}{\lambda_{ik}} e_{ik}, \\ \mathrm{ad}^\star_{e_{ij}} e_{ik} &= -\frac{\lambda_{ik}}{\lambda_{jk}} e_{jk} \quad \mathrm{ad}^\star_{e_{ik}} e_{jk} = -\frac{\lambda_{jk}}{\lambda_{ij}} e_{ij} \quad \mathrm{ad}^\star_{e_{jk}} e_{ij} = -\frac{\lambda_{ij}}{\lambda_{ik}} e_{ik} \end{aligned} \tag{5.3}$$

and all others are zero.

Our goal is to compute the Ricci curvature in this orthogonal basis, and we recall that for any vectors v and w, we have

$$\mathrm{Ric}(v, w) = \sum_{k<\ell} \frac{g\big(R(e_{k\ell}, v)w, e_{k\ell}\big)}{\lambda_{k\ell}}.$$

Polarizing formula (2.9) for the curvature, fixing u, we have

$$\begin{aligned} g(R(u,v)w,u) = \frac{1}{4} g(\mathrm{ad}^\star_u v + \mathrm{ad}^\star_v u + \mathrm{ad}_u v, \mathrm{ad}^\star_u w + \mathrm{ad}^\star_w u + \mathrm{ad}_u w) - \tfrac{1}{2} g(\mathrm{ad}^\star_u v + \mathrm{ad}_u v, \mathrm{ad}_u w) \\ - \tfrac{1}{2} g(\mathrm{ad}^\star_u w + \mathrm{ad}_u w, \mathrm{ad}_u v) - \tfrac{1}{2} g(\mathrm{ad}^\star_u u, \mathrm{ad}^\star_v w + \mathrm{ad}^\star_w v). \end{aligned} \tag{5.4}$$

Thus when we consider $u = e_{k\ell}$, the last term will be zero since $\mathrm{ad}^\star_u u = 0$. Furthermore if $v = e_{ij}$, then by formulas (5.2) and (5.3), the only way to have $\mathrm{ad}^\star_u v$, $\mathrm{ad}^\star_v u$, or $\mathrm{ad}_u v$ nonzero is if $\{i,j\} \cap \{k,\ell\}$ has exactly one element, and all of those terms are proportional to the same basis element. Thus the only way to have either of these inner products nonzero is if w is the same basis element as v. Since this is true for every $u = e_{k\ell}$, we conclude that the Ricci curvature is diagonal in the basis $\{e_{ij}\}$.

When performing the sum with fixed $v = e_{ij}$ for $i<j$, it is therefore sufficient to compute only terms of the form $g(R(u,v)v,u)$ when either $k<i<j$ with $u = e_{ki}$ or $u = e_{kj}$; or $i<k<j$ with $u = e_{ik}$ or $u = e_{kj}$; or $i<j<k$ with $u = e_{ik}$ or $u = e_{jk}$. The computations are similar in the three ranges for k, and we end up with the same answer, so we will only present this last case.

The Ricci curvature is given in the orthogonal basis by

$$\operatorname{Ric}(e_{ij}, e_{ij}) = \sum_{k\neq i,j} \frac{1}{\lambda_{ik}} g\big(R(e_{ij}, e_{ik})e_{ik}, e_{ij}\big) + \sum_{k\neq i,j} \frac{1}{\lambda_{jk}} g\big(R(e_{ij}, e_{jk})e_{jk}, e_{ij}\big). \tag{5.5}$$

Fixing $k > j$ and using (2.9), we get

$$\begin{aligned} g\big(R(e_{ij}, e_{ik})e_{ik}, e_{ij}\big) &= \tfrac{1}{4}\big\|\operatorname{ad}^\star_{e_{ij}} e_{ik} + \operatorname{ad}^\star_{e_{ik}} e_{ij} + [e_{ij}, e_{ik}]\big\|^2 - g\big(\operatorname{ad}^\star_{e_{ij}} e_{ik} + [e_{ij}, e_{ik}], [e_{ij}, e_{ik}]\big) \\ &= \frac{(\lambda_{ij} + \lambda_{jk} - \lambda_{ik})^2}{4\lambda_{jk}} + \lambda_{ik} - \lambda_{jk}. \end{aligned}$$

Similarly we get

$$g\big(R(e_{ij}, e_{jk})e_{jk}, e_{jk}\big) = \frac{(-\lambda_{ij} + \lambda_{jk} - \lambda_{ik})^2}{4\lambda_{ik}} + \lambda_{jk} - \lambda_{ik}.$$

Combining these, we thus have that the k term in (5.5) simplifies to

$$\frac{g\big(R(e_{ij}, e_{ik})e_{ik}, e_{ij}\big)}{\lambda_{ik}} + \frac{g\big(R(e_{ij}, e_{jk})e_{jk}, e_{ij}\big)}{\lambda_{jk}} = \frac{(\lambda_{ij} - \lambda_{ik} + \lambda_{jk})(\lambda_{ij} + \lambda_{ik} - \lambda_{jk})}{2\lambda_{ik}\lambda_{jk}}.$$

The terms with $k < i$ and $i < k < j$ are similar. □

5.2 Steady geodesics on $SO(n)$

Using Theorem 4.2, we show that although the Ricci curvature of the generalized rigid body metric (5.1) can be negative, we still get conjugate points along steady geodesics, just like for the standard rigid body metric. As pointed out in Remark 4.3, some of these conjugate points are not detected by the Misiołek criterion.

Proposition 5.4. *If $u_0 = e_{ij}$ is a steady solution of the Euler equation for the generalized rigid body metric* (5.1)*, then there are conjugate points along the corresponding steady geodesic.*

Proof. According to Lemma 5.1, the basis vectors $e_{k\ell}$ for which $\operatorname{ad}_{e_{ij}}$ is not zero are the ones where $\{k, \ell\} \cap \{i, j\} \neq \emptyset$, i.e. $2m$ vectors in total with $m := n - 2$. We can then reorder and rename the basis vectors $\{e_{ij}\}_{1\leq i<j\leq n}$ into $\{w_k\}_{1\leq k\leq n(n-1)/2}$ so that

$$[u_0, w_{2j-1}] = w_{2j}, \quad [w_{2j-1}, w_{2j}] = u_0, \quad [w_{2j}, u_0] = w_{2j-1}, \quad 1 \leq j \leq m,$$

which yields $L(w_{2j-1}) = w_{2j}$, $L(w_{2j}) = -w_{2j-1}$ for $1 \leq j \leq m$, and $L(w_k) = 0$ for all other k. Thus the matrix representation of $L = \operatorname{ad}_{u_0}$ in the basis $\{w_i\}_{1\leq i\leq 2m}$ is a $(2m) \times (2m)$ matrix composed of 2×2 nonzero blocks on the diagonal, all equal to

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and zero everywhere else. Therefore the matrix representation of L^2 is the identity, which obviously commutes with Λ. This proves the existence of conjugate points by Theorem 4.2. □

5.3 Nonsteady geodesics on $SO(3)$

In this section, we show a concrete example of application of Theorem 1.2 for nonsteady geodesics. We consider dimension $n = 3$ and the standard rigid body metric, i.e., metric (5.1) with $\Lambda(u) = \frac{1}{2}(Mu + uM)$ where M is a diagonal matrix with diagonal entries $\mu_1, \mu_2, \mu_3 > 0$. The computations depend on the following quantities, which we set to specific values for the sake of simplicity:

$$\frac{\mu_2 + \mu_3}{2} = 4, \quad \frac{\mu_1 + \mu_3}{2} = 3, \quad \frac{\mu_1 + \mu_2}{2} = 2.$$

Let $\eta(t)$ be a nonsteady geodesic on $SO(3)$. Taking advantage of the fact that on $\mathfrak{so}(3)$, the Lie bracket of antisymmetric matrices is exactly the cross product of the corresponding vectors, we can write the Euler-Arnold equation (2.14) in the form

$$\eta' = \eta u, \qquad \Lambda u' = (\Lambda u) \times u. \tag{5.6}$$

Decomposing the Lie algebra velocity in the canonical basis $\{e_{23}, e_{13}, e_{12}\}$,

$$u(t) = p(t)e_{23} + q(t)e_{13} + r(t)e_{12},$$

we can write the Euler-Arnold equation as

$$p' = \frac{1}{4}qr, \quad q' = -\frac{2}{3}pr, \quad r' = \frac{1}{2}pq. \tag{5.7}$$

These equations were solved explicitly by Jacobi [29] in terms of what are now known as Jacobi elliptic functions.

One can show that the quantities involved in the criterion of Theorem 1.2 are given by

$$\zeta(t) = \left(\tfrac{1}{3}kp(t)^2 + \rho\right)^{-1/2}, \quad \phi(t) = \tfrac{1}{72}\ell\zeta(t)^4\big((9k-2\ell)p(t)^2+3\rho\big) \quad \text{with } \rho := \tfrac{1}{24}(3k-\ell)(\ell-2k).$$

Recall that conjugate points occur if one of the two following conditions is satisfied: (i) the functions ζ and ϕ are bounded respectively above and below by positive constants, or (ii) the quantity $\psi = \frac{\phi}{\zeta^2} - \frac{\zeta''}{\zeta}$ is bounded below by a positive number. Using the conservation laws (3.2)

$$k = 4p(t)^2 + 3q(t)^2 + 2r(t)^2, \qquad \ell = 16p(t)^2 + 9q(t)^2 + 4r(t)^2,$$

we see that $4k - \ell > 0$, $2k - \ell < 0$, while $3k - \ell$ can take both signs.

If $3k > \ell$ then $\rho > 0$ and we immediately obtain the upper bound $\zeta < \rho^{-1/2}$. We can also obtain a lower bound for ϕ and apply Theorem 1.2 under the condition (i). The case $3k < \ell$ can be dealt with in the same way, expressing everything in terms of $r(t)$ instead of $p(t)$.

In the limit case $3k = \ell$, we do not have an upper bound for ζ since $\rho = 0$, and we need the second version of Theorem 1.2, i.e., the condition on ψ. In this particular case, we have $r(t) = \pm\sqrt{2}p(t)$, so that the Euler-Arnold equations (5.7) become

$$p' = \pm\tfrac{\sqrt{2}}{4}qp, \qquad q' = \mp\tfrac{2\sqrt{2}}{3}p^2.$$

The solutions are given by

$$p(t) = \pm\sqrt{3}m \operatorname{sech} m(t - t_0), \qquad q(t) = \pm 2\sqrt{2}m \tanh m(t - t_0),$$

for any constant m and any t_0. However the constraint $\langle \Lambda u, \Lambda u\rangle = \ell$ implies that

$$\ell = 24p^2 + 9q^2 = 72m^2\left(\operatorname{sech}^2\big(m(t-t_0)\big) + \tanh^2\big(m(t-t_0)\big)\right) = 72m^2,$$

so that $m^2 = \ell/72$. This yields

$$\frac{\zeta''}{\zeta} = 2\left(\frac{p'}{p}\right)^2 - \frac{p''}{p} = m^2 = \frac{\ell}{72} \quad \text{and} \quad \frac{\phi(t)}{\zeta(t)^2} = \frac{\ell\zeta(t)^2(9k-2\ell)p(t)^2}{72} = \frac{\ell}{8},$$

and so finally $\psi(t) = \ell/9$ is a positive constant and condition (ii) of Theorem 1.2 is trivially satisfied.

We have seen that in the n-dimensional rigid body metric, every geodesic eventually has a conjugate point, since the Ricci curvature is strictly positive. For the generalized rigid body metric, we know that all steady geodesics experience conjugate points, but it is unclear whether all nonsteady geodesics experience them, and we do not have a counterexample.

6 Berger-Cheeger groups

An interesting class of quadratic groups where the Euler-Arnold equation (1.1) may be solved explicitly to obtain a family of nonsteady solutions is given by the following construction due originally to Berger [6] (known as the Berger spheres when $G = S^3$ and $H = S^1$) and later generalized by Cheeger [11].

Definition 6.1. *Consider a Lie group G endowed with a positive-definite bi-invariant metric $\langle\cdot,\cdot\rangle$. In particular, G must be a Cartesian product of a compact group and an abelian group. Suppose H is a subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the respective Lie algebras of G and H. The Cheeger deformation along H is defined to be a left-invariant metric g on G, given by*

$$g(u,v) = \langle u, v\rangle + \delta\langle Pu, Pv\rangle, \tag{6.1}$$

where $\delta \in \mathbb{R}$ is a parameter larger than -1, and P is the orthogonal projection onto $\mathfrak{h} \subset \mathfrak{g}$. We denote the associated symmetric operator $\Lambda = I + \delta P$.

The curvatures of such metrics have been explored in some detail, and some general facts about (6.1) are known: for $-1 < \delta < 0$, we are shrinking the metric along the subalgebra $\mathfrak{h}$, and this is known to preserve nonnegative sectional curvature, whereas for $\delta > 0$ curvature can become negative (see [24], [43]). The same technique can be used to get manifolds of positive Ricci curvature [34].

We obtain several new results about metrics of the form (6.1).

First, we derive an explicit formula for its geodesics in terms of the Lie group exponential. Using this formula and Theorem 1.1, we get a fairly simple criterion for conjugate points along nonsteady geodesics.

We also apply Theorem 1.2 to this case, and show that the function $\phi(t)$ in that theorem reduces to a constant, which can be easily computed for any initial velocity $u_0 \in \mathfrak{g}$. If this constant is positive, then the geodesic with initial velocity u_0 will eventually develop conjugate points.

Finally, we show that if the extra condition $[\mathfrak{h}^\perp, \mathfrak{h}^\perp] \subseteq \mathfrak{h}$ is imposed, the Ricci curvature of (6.1) is "block Einstein," which means that it becomes a multiple of the identity when restricted to $\mathfrak{h}$, and a different multiple of the identity when restricted to $\mathfrak{h}^\perp$.

An unexpected feature of these metrics is that despite the changes in curvature as δ varies, even to the extent of producing metrics with some negative Ricci curvatures, the behavior of geodesics and conjugate points remains essentially the same.

6.1 Geodesic equation and conjugate points

We begin with the solution of the Euler-Arnold equation.

Proposition 6.2. *In the same notation as Definition 6.1, let $\mathfrak{h}^\perp$ denote the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ under the bi-invariant metric $\langle\cdot,\cdot\rangle$. Then*

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h} \qquad \text{and} \qquad [\mathfrak{h}, \mathfrak{h}^\perp] \subset \mathfrak{h}^\perp. \tag{6.2}$$

As a consequence, if we write $u(t) = p(t) + q(t)$ where $p \in \mathfrak{h}$ and $q \in \mathfrak{h}^\perp$, then the Euler-Arnold equation (2.14) *for u becomes*

$$p'(t) = 0, \qquad \frac{dq}{dt} = \delta \mathrm{ad}_{p(t)} q(t), \tag{6.3}$$

with solution

$$p(t) = p_0, \qquad q(t) = \mathrm{Ad}_{\eta(t)} q_0, \qquad \eta(t) := \exp(\delta t p_0).$$

Proof. The first part of (6.2) is just the definition of a Lie subalgebra. To show the second part, let $u \in \mathfrak{h}$ and $w \in \mathfrak{h}^\perp$: we will show that $\langle \mathrm{ad}_u w, v\rangle = 0$ for any $v \in \mathfrak{h}$. This follows from ad-invariance of the metric and $\mathrm{ad}_u v \in \mathfrak{h}$, since

$$\langle \mathrm{ad}_u w, v\rangle = -\langle w, \mathrm{ad}_u v\rangle = 0.$$

Writing $u = p + q$ where $p \in \mathfrak{h}$ and $q \in \mathfrak{h}^\perp$, we obtain $\Lambda u = (1+\delta)p + q$, so that

$$\mathrm{ad}_u \Lambda u = \mathrm{ad}_{p+q}\big((1+\delta)p + q\big) = \mathrm{ad}_p q + (1+\delta)\mathrm{ad}_q p = -\delta \mathrm{ad}_p q.$$

The Euler-Arnold equation (2.14) thus becomes

$$(1+\delta)p'(t) + q'(t) - \delta \mathrm{ad}_{p(t)} q(t) = 0,$$

and since $\mathrm{ad}_p q \in \mathfrak{h}^\perp$, we obtain the splitting (6.3). Obviously $p(t) = p_0$ solves the first part, and the second part follows from the definition of ad as the derivative of Ad on any group. □

Geodesics can be described explicitly in terms of the group exponential map as follows.

Proposition 6.3. *In the same notation as Definition 6.1, if $\gamma(t)$ is a geodesic with $\gamma(0) = \mathrm{id}$ and $\gamma'(0) = u_0 = p_0 + q_0$, for $p_0 \in \mathfrak{h}$ and $q_0 \in \mathfrak{h}^\perp$, then*

$$\gamma(t) = e^{t\Lambda u_0} e^{-\delta t p_0}. \tag{6.4}$$

Proof. In Proposition 6.2, we found that the solution $u(t)$ of the Euler-Arnold equation (2.14) was given by

$$u(t) = e^{\delta t p_0}(p_0 + q_0)e^{-\delta t p_0} = \eta(t) u_0 \eta(t)^{-1}.$$

Now using the flow equation $\gamma'(t) = \gamma(t)u(t)$, we find that

$$\begin{aligned} \frac{d}{dt}\big(\gamma(t)\eta(t)\big) &= \gamma'(t)\eta(t) + \gamma(t)\eta'(t) = \gamma(t)\eta(t)u_0\eta(t)^{-1}\eta(t) + \delta\gamma(t)\eta(t)p_0 \\ &= \gamma(t)\eta(t)(u_0 + \delta p_0) = \gamma(t)\eta(t)\Lambda u_0. \end{aligned}$$

Thus since $\gamma(0)\eta(0) = \mathrm{id}$, we must have

$$\gamma(t)\eta(t) = e^{t\Lambda u_0},$$

and formula (6.4) follows using $\eta(t) = e^{\delta t p_0}$. □

This formula together with Theorem 1.1 gives a simple criterion for conjugate points along nonsteady geodesics in any Berger-Cheeger group. Essentially, if initial velocity Λu_0 would yield a closed geodesic under the bi-invariant metric on G, then initial velocity u_0 will yield a geodesic with conjugate points under the Berger-Cheeger metric. Since closed geodesics for bi-invariant metrics on compact Lie groups are common, this yields many examples.

Corollary 6.4. *Suppose G is a Berger-Cheeger group, and γ is a nonsteady geodesic with $\gamma(0) = \mathrm{id}$ and $\gamma'(0) = u_0$. Assume that $e^{\tau\Lambda u_0} = \mathrm{id}$ for some $\tau > 0$. Then there is a conjugate point along the geodesic γ given by* (6.4).

Proof. By the inclusions (6.2), the operators ad_{p_0} for $p_0 \in \mathfrak{h}$ preserve the orthogonal decomposition, and therefore so does $\mathrm{Ad}_{\eta(t)}$ for any t; in particular $\mathrm{Ad}_{\eta(t)}$ commutes with $\Lambda = I + \delta P$. We conclude that $\mathrm{Ad}_{\eta(t)}$ is an isometry of the Berger-Cheeger metric, since it is an isometry of the bi-invariant metric.

By the explicit formula (6.4), we will have $\gamma(\tau) = \eta(\tau)^{-1}$. Thus $\mathrm{Ad}_{\gamma(\tau)}$ preserves the Berger-Cheeger metric as well, and we conclude that right-translation by $\gamma(\tau)$ is an isometry. Thus Theorem 1.1 implies that $\gamma(\tau)$ is conjugate to $\gamma(0)$. □

In the next theorem, we show how Theorem 1.2 applies in the class of Berger-Cheeger groups. Throughout the proof, we use the terminology introduced in Theorem 1.2, directing the reader to Section 3.2 for detailed definitions of these terms.

Theorem 6.5. *Let G be a Berger-Cheeger group with left-invariant metric defined by* (6.1) *with parameter δ. For $-1 < \delta < 0$, every nonsteady geodesic has conjugate points.*

On the other hand, if $\delta > 0$ and $u_0 = p_0 + q_0$ is the initial condition of a nonsteady geodesic γ with $p_0 \in \mathfrak{h}$ and $q_0 \in \mathfrak{h}^\perp$, then γ develops conjugate points if either δ is sufficiently small (the precise value depending on u_0), or if $|p_0|$ is sufficiently large compared to $|q_0|$.

Proof. First we show that $g(u'(t), u'(t))$ is constant, which then yields that $\zeta(t)$ is also constant. We have

$$u'(t) = q'(t) = \delta \mathrm{ad}_{p_0} q(t) = \delta \mathrm{Ad}_{\eta(t)}[p_0, q_0].$$

As in the proof of Corollary 6.4, $\mathrm{Ad}_{\eta(t)}$ commutes with Λ and thus preserves both the bi-invariant metric and the Berger-Cheeger metric. We therefore have

$$g(u'(t), u'(t)) = \langle q'(t), q'(t)\rangle = \delta^2 \big|\mathrm{Ad}_{\eta(t)}[p_0, q_0]\big|^2 = \delta^2 |[p_0, q_0]|^2,$$

where $|\ |$ denotes the norm with respect to $\langle\cdot,\cdot\rangle$. Thus, $\zeta(t) = (\delta|[p_0, q_0]|)^{-1}$ which is constant. Writing $\zeta(t) = \zeta_0$, we have $v_1(t) = \zeta_0^2 u'(t) = \zeta_0^2 \delta \mathrm{Ad}_{\eta(t)}[p_0, q_0]$.

In particular, showing that the function $\psi(t)$ in Theorem 1.2 is bounded below by a positive constant reduces to showing the same for $\phi(t)$, which we recall is given by

$$\phi(t) := \frac{k^2 \langle \Lambda w, \Lambda u\rangle^2}{g(v_2, v_2)} - \langle w, x\rangle.$$

A long but straightforward computation shows that the terms appearing in $\phi(t)$ can all be written explicitly in terms of p_0, q_0 and δ as follows.

$$\begin{aligned}
k &= \langle u(t), \Lambda u(t)\rangle = (1+\delta)|p_0|^2 + |q_0|^2,\\
\ell &= \langle u(t), \Lambda^2 u(t)\rangle = (1+\delta)^2|p_0|^2 + |q_0|^2\\
v_2(t) &= \delta\big(|q_0|^2 p_0 - (1+\delta)|p_0|^2 q(t)\big),\\
g(v_2(t), v_2(t)) &= \delta^2(1+\delta)|p_0|^2|q_0|^2 k,\\
w(t) &= \delta\zeta_0^2\, \mathrm{Ad}_{\eta(t)}\big[(1+\delta)p_0 + q_0, [p_0, q_0]\big],\\
x(t) &= \delta^2\zeta_0^2\, \mathrm{Ad}_{\eta(t)} P\big([q_0, [p_0, q_0]]\big).
\end{aligned}$$

We point out that in the computation of $x(t)$, we used the fact that $P([p_0, [p_0, q_0]]) = 0$ from (6.2) to simplify some terms. From the last two lines above, we obtain

$$\langle w, x\rangle = \delta^3\zeta_0^4 \big|P\big([q_0, [p_0, q_0]]\big)\big|^2,$$

using the fact that P is an orthogonal projection. In addition, one can show that

$$\langle \Lambda w, \Lambda u\rangle = \delta^2(1+\delta)\zeta_0^2 \big|[p_0, q_0]\big|^2.$$

Putting together the above computations, we arrive at the following formula for $\phi(t)$, which turns out to be a constant, depending only on the initial condition and the parameter $\delta > -1$.

$$\phi = \delta^2\zeta_0^4 \left(\frac{(1+\delta)\big|[p_0, q_0]\big|^4 \big((1+\delta)|p_0|^2 + |q_0|^2\big)}{|p_0|^2|q_0|^2} - \delta \big|P\big([q_0, [p_0, q_0]]\big)\big|^2 \right)$$

Clearly, for $-1 < \delta \le 0$, the above quantity is strictly positive, so conjugate points always develop along nonsteady geodesics. On the other hand, even when $\delta > 0$, in which case we no longer have positive curvature, ϕ is still positive for fixed u_0 and sufficiently small δ. Furthermore, given any $\delta > 0$, rescaling q_0 to cq_0, we get

$$\phi = \delta^2 c^2 \zeta_0^4 \left(\frac{(1+\delta)\big|[p_0, q_0]\big|^4 \big((1+\delta)|p_0|^2 + c^2|q_0|^2\big)}{|p_0|^2|q_0|^2} - \delta c^2 \big|P\big([q_0, [p_0, q_0]]\big)\big|^2 \right),$$

which is clearly positive for sufficiently small c. Geometrically, this means that all nonsteady geodesics which are sufficiently close to the subgroup H will develop conjugate points, regardless of what happens to curvature. □

We apply Theorem 6.5 to the Zeitlin model, discussed in Section 2. The very first of these groups is the eight-dimensional group $SU(3)$ equipped with the Zeitlin metric, which can be described as the Berger-Cheeger metric obtained from taking $G = SU(3)$, $H = SO(3)$ and $\delta = -2/3$ in Definition 6.1. In particular, since δ is negative, we immediately obtain the following corollary of Theorem 6.5.

Corollary 6.6. *Every nonsteady geodesic in the $SU(3)$ Zeitlin model has a conjugate point.*

6.2 Ricci curvature

In this section, we compute the Ricci curvature of Berger-Cheeger groups. In the special case that $\mathfrak{h}$ and $\mathfrak{h}^\perp$ factor $\mathfrak{g}$ into a Cartan decomposition, which requires the additional assumption $[\mathfrak{h}^\perp, \mathfrak{h}^\perp] \subseteq \mathfrak{h}$, we get a substantial simplification of the Ricci curvature: the group becomes "block Einstein" in the sense that the Ricci curvature is a constant multiple of the metric when restricted to $\mathfrak{h}$, and a different constant multiple of the metric when restricted to $\mathfrak{h}^\perp$. We focus on this case, and provide a general formula without this assumption at the end – see Remark 6.9.

Proposition 6.7. *Suppose G is a Berger-Cheeger group with subgroup H. Assume that*

$$[\mathfrak{h}^\perp, \mathfrak{h}^\perp] \subseteq \mathfrak{h}. \tag{6.5}$$

Let P denote the orthogonal projection of $\mathfrak{g}$ onto $\mathfrak{h}$ (in the bi-invariant form on $\mathfrak{g}$), and let $Q = I - P$ be the projection onto $\mathfrak{h}^\perp$. If $u \in \mathfrak{h}$ and $v \in \mathfrak{g}$, then the curvature of G under the left-invariant metric (6.1) *is given by*

$$g(R(u,v)v,u) = \frac{1+\delta}{4}\big|\mathrm{ad}_u P(v)\big|^2 + \frac{(1+\delta)^2}{4}\big|\mathrm{ad}_u Q(v)\big|^2. \tag{6.6}$$

Meanwhile if $u \in \mathfrak{h}^\perp$ and $v \in \mathfrak{g}$, then

$$g(R(u,v)v,u) = \frac{(1+\delta)^2}{4}\left|\mathrm{ad}_u P(v)\right|^2 + \frac{1-3\delta}{4}\left|\mathrm{ad}_u Q(v)\right|^2. \tag{6.7}$$

As before, for any $w \in \mathfrak{g}$, we write $|w|^2 = \langle w, w\rangle$ while $\|w\|^2 = g(w,w) = \langle w, \Lambda w\rangle$.

Proof. We begin with the general formula for sectional curvature on a Lie group with left-invariant metric g as in [32]:

$$g\big(R(u,v)v,u\big) = \tfrac{1}{4}\|\mathrm{ad}_u^\star v + \mathrm{ad}_v^\star u + \mathrm{ad}_u v\|^2 - g(\mathrm{ad}_u^\star v + \mathrm{ad}_u v, \mathrm{ad}_u v) - g(\mathrm{ad}_u^\star u, \mathrm{ad}_v^\star v). \tag{6.8}$$

Recalling the formula (2.13) for $\mathrm{ad}^\star$, and using the formula $\Lambda = I + \delta P$ and its consequence $\Lambda^{-1} = I - \frac{\delta}{1+\delta}P$, we find that for any u and v, we have

$$\mathrm{ad}_u^\star v = -\mathrm{ad}_u v - \delta \mathrm{ad}_u P(v) + \tfrac{\delta}{1+\delta} P(\mathrm{ad}_u v) + \tfrac{\delta^2}{1+\delta} P\big(\mathrm{ad}_u P(v)\big),$$

and we conclude, using the commutator relations (6.2) and (6.5) between $\mathfrak{h}$ and $\mathfrak{h}^\perp$, that

$$\mathrm{ad}_u^\star v + \mathrm{ad}_u v = \delta\left(\mathrm{ad}_{Q(u)}\left(\frac{Q(v)}{1+\delta} - P(v)\right)\right), \tag{6.9}$$

where as expected if $\delta = 0$ the right side is zero since then the metric becomes bi-invariant on $\mathfrak{g}$. If $u \in \mathfrak{h}$, then clearly (6.9) is zero, and together with $\mathrm{ad}_u^\star u = 0$ this shows that the last two terms in (6.8) disappear. Decomposing v into $P(v) + Q(v)$ immediately gives (6.6).

Now suppose that $u \in \mathfrak{h}^\perp$. Using (6.9), after simplifying we get

$$\begin{aligned}\mathrm{ad}_u^\star v + \mathrm{ad}_u v + \mathrm{ad}_v^\star u &= \mathrm{ad}_u Q(v) + (1-\delta)\mathrm{ad}_u P(v),\\ g\left(\mathrm{ad}_u^\star v + \mathrm{ad}_u v, \mathrm{ad}_u v\right) &= \delta\left(|\mathrm{ad}_u Q(v)|^2 - |\mathrm{ad}_u P(v)|^2\right).\end{aligned}$$

Thus, formula (6.8) becomes

$$\begin{aligned}\langle R(u,v)v,u\rangle &= \frac{1+\delta}{4}\,|\mathrm{ad}_u Q(v)|^2 + \frac{(1-\delta)^2}{4}\,|\mathrm{ad}_u P(v)|^2 - \delta\left(|\mathrm{ad}_u Q(v)|^2 - |\mathrm{ad}_u P(v)|^2\right)\\ &= \frac{1-3\delta}{4}\,|\mathrm{ad}_u Q(v)|^2 + \frac{(1+\delta)^2}{4}\,|\mathrm{ad}_u P(v)|^2,\end{aligned}$$

which is (6.7). □

Theorem 6.8. *Suppose G is a compact simple Lie group and H is a compact simple Lie subgroup, with respective Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, and consider the metric g given by* (6.1)*. Then the Ricci curvature of G under g splits into block diagonal form as*

$$\mathrm{Ric}(v,v) = C_1(\delta)|P(v)|^2 + C_2(\delta)|Q(v)|^2,$$

for some constants C_1 and C_2 depending on G, H and the parameter δ.

Proof. In the bi-invariant metric $\langle\cdot,\cdot\rangle$, construct an orthonormal basis $e_1,\dots,e_m$ of $\mathfrak{h}$ and an orthonormal basis $f_1,\dots,f_n$ of $\mathfrak{h}^\perp$. Write $v = x + y$, where $x \in \mathfrak{h}$ and $y \in \mathfrak{h}^\perp$. Then, formulas (6.6) and (6.7) become

$$\begin{aligned}g\big(R(e_i,v)v,e_i\big) &= \frac{1+\delta}{4}|\mathrm{ad}_{e_i}x|^2 + \frac{(1+\delta)^2}{4}|\mathrm{ad}_{e_i}y|^2, \qquad 1 \le i \le m,\\ g\big(R(f_j,v)v,f_j\big) &= \frac{(1+\delta)^2}{4}\,|\mathrm{ad}_{f_j}x|^2 + \frac{1-3\delta}{4}\,|\mathrm{ad}_{f_j}y|^2, \qquad 1 \le j \le n.\end{aligned} \tag{6.10}$$

Note that the vectors $\{e_1,\dots,e_m,f_1,\dots,f_n\}$ form an orthonormal basis in the bi-invariant metric, but not in the g metric; instead the vectors $\{(1+\delta)^{-1/2}e_1,\dots,(1+\delta)^{-1/2}e_m,f_1,\dots,f_n\}$ are g-orthonormal. Thus by (6.10), the Ricci curvature is

$$\begin{aligned}\mathrm{Ric}(v,v) &= \frac{1}{1+\delta}\sum_{i=1}^m g\big(R(e_i,v)v,e_i\big) + \sum_{j=1}^n g\big(R(f_j,v)v,f_j\big)\\ &= \frac{1}{4}\sum_{i=1}^m|\mathrm{ad}_{e_i}x|^2 + \frac{1+\delta}{4}\sum_{i=1}^m|\mathrm{ad}_{e_i}y|^2 + \frac{1-3\delta}{4}\sum_{j=1}^n|\mathrm{ad}_{f_j}y|^2 + \frac{(1+\delta)^2}{4}\sum_{j=1}^n|\mathrm{ad}_{f_j}x|^2.\end{aligned} \tag{6.11}$$

We conclude that the Ricci curvature is given for $v = x + y$ by

$$\mathrm{Ric}(v,v) = \mathrm{Ric}(x,x) + \mathrm{Ric}(y,y)$$

since there are no cross-terms in (6.11), and thus it is sufficient to compute them separately. The claim of the theorem now reduces to showing that $\mathrm{Ric}(x,x) = C_1|x|^2$ and $\mathrm{Ric}(y,y) = C_2|y|^2$ for $x \in \mathfrak{h}$ and $y \in \mathfrak{h}^\perp$.

The main principle we will use is that since G is a compact simple Lie group, there is a unique non-degenerate bi-invariant metric on G (up to a constant multiple), and thus there is a positive constant β_G such that for any $v \in \mathfrak{g}$, we have

$$\mathrm{Tr}(\mathrm{ad}_v \mathrm{ad}_v) = -\beta_G |v|^2.$$

See for example [1], Proposition 2.48. This says more explicitly that

$$\sum_{i=1}^m |\mathrm{ad}_{e_i} v|^2 + \sum_{j=1}^n |\mathrm{ad}_{f_j} v|^2 = \beta_G |v|^2.$$

This formula applies for any v, but we also have a compact simple Lie group H, and the same principle applies to give

$$\sum_{i=1}^m |\mathrm{ad}_{e_i} x|^2 = \beta_H |x|^2,$$

where the sum is only taken over $\mathfrak{h}$ and only applied to $x \in \mathfrak{h}$. The Ricci curvature on x is now given using (6.11) by

$$\begin{aligned}
\mathrm{Ric}(x,x) &= \frac{1}{4}\sum_{i=1}^m |\mathrm{ad}_{e_i} x|^2 + \frac{(1+\delta)^2}{4}\sum_{j=1}^n |\mathrm{ad}_{f_j} x|^2 \\
&= \frac{(1+\delta)^2}{4}\left(\sum_{i=1}^m |\mathrm{ad}_{e_i} x|^2 + \sum_{j=1}^n |\mathrm{ad}_{f_j} x|^2\right) - \frac{\delta(2+\delta)}{4}\sum_{i=1}^m |\mathrm{ad}_{e_i} x|^2 \\
&= \frac{(1+\delta)^2\beta_G}{4}|x|^2 - \frac{\delta(2+\delta)\beta_H}{4}|x|^2,
\end{aligned}$$

and we conclude that

$$C_1(\delta) = \frac{(1+\delta)^2\beta_G - \delta(2+\delta)\beta_H}{4}.$$

The computation for $\mathrm{Ric}(y,y)$ is slightly more complicated since $\mathfrak{h}^\perp$ is not a Lie subalgebra, and thus the sum of terms $|\mathrm{ad}_{e_i} y|^2$ does not represent a Killing form. However we have by (6.11) that

$$\mathrm{Ric}(y,y) = \frac{1+\delta}{4}\sum_{i=1}^m |\mathrm{ad}_{e_i} y|^2 + \frac{1-3\delta}{4}\sum_{j=1}^n |\mathrm{ad}_{f_j} y|^2. \tag{6.12}$$

Now to compute these two sums, we use the fact that $\mathrm{ad}_{e_i} y \in \mathfrak{h}^\perp$ by (6.2) to write

$$\sum_{i=1}^m |\mathrm{ad}_{e_i} y|^2 = \sum_{i=1}^m \sum_{j=1}^n \langle \mathrm{ad}_{e_i} y, f_j\rangle^2,$$

summing only over the inner products with vectors $f_j \in \mathfrak{h}^\perp$, and similarly since $\mathrm{ad}_{f_j} y \in \mathfrak{h}$ by our additional assumption (6.5),

$$\sum_{j=1}^n |\mathrm{ad}_{f_j} y|^2 = \sum_{j=1}^n \sum_{i=1}^m \langle \mathrm{ad}_{f_j} y, e_i\rangle^2.$$

We now notice that these two sums are exactly the same, since

$$\langle \mathrm{ad}_{f_j} y, e_i\rangle = -\langle y, \mathrm{ad}_{f_j} e_i\rangle = \langle y, \mathrm{ad}_{e_i} f_j\rangle = -\langle \mathrm{ad}_{e_i} y, f_j\rangle,$$

and thus we get

$$\sum_{i=1}^{m} |\mathrm{ad}_{e_i} y|^2 + \sum_{j=1}^{n} |\mathrm{ad}_{f_j} y|^2 = 2\sum_{j=1}^{n} |\mathrm{ad}_{f_j} y|^2 = \beta_G |y|^2.$$

Formula (6.12) thus becomes

$$\mathrm{Ric}(y,y) = \frac{(1+\delta)+(1-3\delta)}{4} \sum_{j=1}^{n} |\mathrm{ad}_{f_j} y|^2 = \frac{(1-\delta)\beta_G |y|^2}{4},$$

so we conclude

$$C_2(\delta) = \frac{(1-\delta)\beta_G}{4}.$$

□

An interesting class of examples where Theorem 6.8 applies is $G = SU(n)$ and $H = SO(n)$, with the natural inclusion. In this case, $\mathfrak{h}$ consists of antisymmetric matrices, and since the bi-invariant form is $\langle u, v\rangle = -\frac{1}{2}\mathrm{Tr}(uv)$, this means that $\mathfrak{h}^\perp$ is a subset of the space of symmetric matrices. Therefore, given $v, w \in \mathfrak{h}^\perp$,

$$[v,w]^T = (vw - wv)^T = w^T v^T - v^T w^T = wv - vw = -[v,w],$$

so that $[v,w] \in \mathfrak{h}$. This shows that the condition $[\mathfrak{h}^\perp, \mathfrak{h}^\perp] \subseteq \mathfrak{h}$ is fulfilled. Here the constants are $\beta_H = 2(n^2 - n - 4)$ (using Proposition 5.2) and $\beta_G = 4n$ (from [1], Example 2.50).

Remark 6.9. *If one does not assume the condition $[\mathfrak{h}^\perp, \mathfrak{h}^\perp] \subseteq \mathfrak{h}$, then Proposition 6.7 still holds, except that in the case $u \in \mathfrak{h}^\perp$, the formula becomes*

$$g(R(u,v)v,u) = \frac{1-3\delta}{4} \left|P\big(\mathrm{ad}_u Q(v)\big)\right|^2 + \frac{1}{4}\left|Q(\mathrm{ad}_u \Lambda v)\right|^2.$$

However this formula alone does not allow us to conclude the Ricci curvature is block-diagonal.

References

[1] M.M. Alexandrino and R.G. Bettiol. *Lie groups and geometric aspects of isometric actions*, Vol. 82. Springer International Publishing, 2015.

[2] V. Arnold, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier (Grenoble) 16 (1966), 319-361.

[3] V. Arnold, B. Khesin, *Topological methods in hydrodynamics*, Second edition, Appl. Math. Sci., 125 Springer, Cham, 2021. xx+455 pp. ISBN:978-3-030-74277-5.

[4] P. Birtea, I. Caşu, T. Ratiu, M. Turhan, *Stability of equilibria for the $\mathfrak{so}(4)$ free rigid body*, J. Nonlinear Sci. (2012) 22 187-212.

[5] J. Benn, *Conjugate points in $D^2_\mu(S^2)$*, J. Geom. Phys. 170 (2021), Paper No. 104369, 14 pp.

[6] M. Berger, *Les variétés Riemanniennes homogénes normales simplement connexes a curbure strictment positive*, Ann. Scuola Norm. Sup. Pisa, 15 (1961), 179-146.

[7] A. Besse, *Manifolds all of Whose Geodesics are Closed*, Ergebisse Grenzgeb. Math. 93. Springer, Berlin (1978).

[8] I. Chavel, *A class of Riemannian homogeneous spaces*, J. Differential Geometry 4 (1970), 13-20.

[9] I. Chavel, *Isotropic Jacobi fields and Jacobi's equation on Riemannian homogeneous spaces*, Comment. Math. Helv., 42, 237-248 (1967).

[10] I. Chavel, *On normal riemannian homogeneous spaces of rank* 1, Bull. Amer. Math. Soc., 73 (1967), 477-481.

[11] J. Cheeger, *Some examples of manifolds of nonnegative curvature*, J. Differential Geometry 8 (1973), 623-628.

[12] L. DeMeyer, *Closed geodesics in compact nilmanifolds*, Manuscripta Math. 105(3), 283-310 (2001). https://doi.org/10.1007/PL00005877

[13] M. do Carmo, *Riemannian geometry*, Math. Theory Appl. Birkhäuser Boston, Inc., Boston, MA, 1992. xiv+300 pp. ISBN:0-8176-3490-8.

[14] T. Drivas, G. Misiołek, B. Shi, T. Yoneda, *Conjugate and cut points in ideal fluid motion* Ann. Math. Québec 46, 1-19 (2021).

[15] P. Eberlein, *Geometry of 2-step nilpotent groups with a left invariant metric*, Ann. Sci. École Norm. Sup. (4) 27(1994), no.5, 611-660.

[16] Y. Eliashberg and T. S. Ratiu, *The diameter of the symplectomorphism group is infinite*, Inventiones mathematicae 103.2 (1991): 327-340.

[17] L. Fehér, I. Marshall, *Stability analysis of some integrable Euler equations for $SO(n)$*, Journal of Nonlinear Mathematical Physics 10 (2003) 304-317.

[18] A. Fomenko, A. Mishchenko, *Euler equations on finite-dimensional Lie groups*, Mathematics of the USSR-Izvestiya 12 (1978) 371–389.

[19] F. Frahm, *Über gewisse differentialgleichungen*, Math. Ann. 8 (1874) 35-44.

[20] W.H. Greub, *Linear Algebra*, 4th edition Springer-Verlag, New York, 1981.

[21] D. Gromoll, W. Meyer, *On complete open manifolds of positive curvature*, Ann. of Math. (2) 90 (1969), 75-90.

[22] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Grad. Stud. Math., 34 American Mathematical Society, Providence, RI, 2001, xxvi+641 pp. ISBN: 0-8218-2848-7.

[23] J. Hoppe and S.-T. Yau. *Some properties of matrix harmonics on* S^2 Comm. Math. Phys 195 (1998): 67-77.

[24] J. Huizenga, K. Tapp, *Invariant metrics with nonnegative curvature on* $SO(4)$ *and other Lie groups*, Michigan Math. J. 55 (2007), no.3, 609-630.

[25] B. Khesin, J. Lenells, G. Misiołek, S.C. Preston, *Curvatures of Sobolev metrics on diffeomorphism groups*, Pure Appl. Math. Q.9 (2013), no.2, 291-332.

[26] J. Itoh, K. Kiyohara, *The cut loci and the conjugate loci on ellipsoids*, Manuscripta Math. 114 (2004), no.2, 247-264.

[27] A. Izosimov, *A note on relative equilibria of a free multidimensional rigid body*, J. Phys. A 45 (2012), no.32, 325203, 6 pp.

[28] A. Izosimov, *Stability of relative equilibria of multidimensional rigid body*, Nonlinearity 27 (2014), no.6, 1419-1443.

[29] C. Jacobi, *Sur la rotation d'un corps*, (French) J. Reine Angew. Math.39 (1850), 293-350.

[30] C. Jacobi, *Vorlesungen über Dynamik*, C.G.J. Jacobi's Gesammelte Werke, 2nd ed., Supplement Volume, Berlin: Georg Reimer, 1884.

[31] A. Knapp, *Lie groups beyond an introduction*, Second edition Progr. Math., 140 Birkhäuser Boston, Inc., Boston, MA, 2002. xviii+812 pp. ISBN:0-8176-4259-5.

[32] J.-M. Lee and S.C. Preston, *Nonpositive curvature of the quantomorphism group and quasigeostrophic motion*, Diff. Geo. Appl. **74** (2021).

[33] K. Lee, K. Park, *Smoothly Closed geodesics in 2-step nilmanifolds* Indiana Univ. Math. J. 45, 1-14 (1996).

[34] L.F. Cavenaghi, R.J.M. e Silva, and L.D. Speran ca, *Positive Ricci curvature through Cheeger deformations*, Collectanea Mathematica, 75(2), 481-510 (2024).

[35] S. Manakov, *Note on the integration of Euler's equations of the dynamics of an n-dimensional rigid body*, Funct. Anal. Appl. (1976) 10 328-9.

[36] M. Mast, *Closed Geodesics in 2-step Nilmanifolds*, Indiana Univ. Math. J. 43, 885-911 (1994).

[37] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Math. 21 (1976), no.3, 293-329.

[38] G. Misiołek, *Conjugate points in $D_\mu(\mathbb{T}^2)$*, Proc. Amer. Math. Soc. 124 (1996), no. 3, 977-982.

[39] K Modin, M. Viviani, *Two-dimensional fluids via matrix hydrodynamics*, arXiv:2405.14282.

[40] S.C. Preston, *Conjugate point criteria on the area-preserving diffeomorphism group*, J. Geom. Phys. 183 (2023), Paper No. 104680, 31 pp.

[41] T. Ratiu, *The motion of the free n-dimensional rigid body*, Indiana Univ. Math. J. (1980) 29 609–29.

[42] H. Rauch, *Geodesics and Jacobi equations on homogeneous Riemannian manifolds*, Proc. United States–Japan Semin. Differ. Geom., Kyoto 1965, Kyoto Univ., Kyoto (1965), pp. 115-127.

[43] L. Schwachhöfer, K. Tapp, *Homogeneous metrics with nonnegative curvature*, J. Geom. Anal. 19 (2009), no.4, 929-943.

[44] N. Smolentsev, *A bi-invariant metric on the group of diffeomorphisms of a three-dimensional manifold*, (Russian) Sibirsk. Mat. Zh. 24 (1983), no.1, 152-159, 193.

[45] V. Zeitlin, *Self-consistent finite-mode approximations for the hydrodynamics of an incompressible fluid on nonrotating and rotating spheres*, Phys. Rev. Lett. 93 (2004), 264501.

[46] W. Ziller, *Closed geodesics on homogeneous spaces*, Math. Z. 152 (1977), no.1, 67-88.

[47] W. Ziller, *The Jacobi equation on naturally reductive compact Riemannian homogeneous spaces*, Comment. Math. Helv., 52, 573-590 (1977).