ISOMETRIC IMMERSIONS AND THE WAVING OF FLAGS

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Abstract. In this article we propose a novel geometric model to study the motion of a physical flag. In our approach a flag is viewed as an isometric immersion from the square with values into $\mathbb{R}^3$ satisfying certain boundary conditions at the flag pole. Under additional regularity constraints we show that the space of all such flags carries the structure of an infinite dimensional manifold and can be viewed as a submanifold of the space of all immersions. The submanifold result is then used to derive the equations of motion, after equipping the space of isometric immersions with its natural kinetic energy. This approach can be viewed in a similar spirit as Arnold's geometric picture for the motion of an incompressible fluid.

1. Introduction

In this article we propose a geometric framework to model the motion of physical flags. Mathematically, a flag on a flagpole may be modeled as an isometric immersion of a square into $\mathbb{R}^3$ subject to the constraint that one edge is mapped to the pole. To obtain the simplest possible model, we ignore external forces and model the flag as though it follows a geodesic in the space of isometric immersions with Riemannian metric determined by the physical kinetic energy. Although the local problem of deformability of isometric immersions is well-known, see e.g., [30] and the references therein, an additional difficulty in our setup consists of the boundary conditions: matching the flag to the pole on one hand, and describing the edges of the square on the other hand. We solve these problems by assuming a relatively simple state of the flag – that it is uniformly furled in one direction – and derive the geodesic equation of motion which is valid as long as the flag remains in this state. Already this situation is surprisingly complicated, and the general situation, e.g., when the flag curves one way and then the other, is left open for future research.

1.1. **Modelling equations of motions as geodesic equations.** Our approach follows similar geometrical models for other situations, such as modeling the motion of ideal fluids as a geodesic evolution in the group of volume-preserving diffeomorphisms, as first done by Arnold [1] in 1966. The advantage of this formulation is that it allows us to relate curvature of the manifold to stability of the system, and that it reduces the system to the

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simplest set of assumptions (without incorporating the details of external forces or the physical composition of the system). Another advantage is that it can lead to rigorous proofs of existence and uniqueness theorems by turning a PDE into an ODE, as in Ebin-Marsden [12] for the incompressible Euler-equation. Many other PDEs have been recast as geodesics in various spaces, most especially on diffeomorphism groups, see e.g., the Korteweg-de Vries equation [20], the Camassa-Holm equation [8, 26, 22], the modified Constantin–Lax–Majda equation [10, 34, 13, 4] or the Hunter-Saxton equation [17, 19, 23, 24]. See [18, 21, 31] for survey articles on the topic and Arnold-Khesin [2] for an introduction to the field and more examples.

Diffeomorphism groups arise in studying motion of fluids which fill up their domain. In many other cases the system is a material moving in a higher-dimensional space, which leads one to work with spaces of embeddings and immersions, see [3] and the references therein. In recent work [28, 29] the third author has analyzed the motion of inextensible threads in Euclidean space: assuming the geometric constraint that the curve η preserves arc length s, we have the equation

$$\frac{\partial^2\eta}{\partial t^2} = \frac{\partial}{\partial s}\Big(\sigma\,\frac{\partial\eta}{\partial s}\Big), \qquad \frac{\partial^2\sigma}{\partial s^2} - \Big|\frac{\partial^2\eta}{\partial s^2}\Big|\sigma = -\Big|\frac{\partial^2\eta}{\partial t\partial s}\Big|^2, \qquad \Big|\frac{\partial\eta}{\partial s}\Big|^2 \equiv 1. \tag{1}$$

This is a very old equation, but the existence theory is rather recent, as is the geometric treatment. A natural extension of this to higher dimensions is to consider the space of embeddings of surfaces into $\mathbb{R}^3$ with some constraint: either preserving the area element or preserving the Riemannian metric. Those that preserve the area element, which serve as a model for the motion of membranes in biological systems, were studied by several researchers including the first author [5, 15, 27]. In this article we study those that preserve the metric, which can serve as a model for unstretchable fabric or paper.

1.2. **Contributions of the article.** One complication in the study of the space of isometric immersions is that this space is relatively small and depends delicately on the geometry of the surface. For example the Cohn-Vossen theorem says that if a surface has nonnegative Gaussian curvature and no open set where the the curvature is zero, then it is rigid: the only deformations are isometries of $\mathbb{R}^3$. For a recent survey of such results, see Han-Hong [16]. If the Gaussian curvature is zero, as in our case, then there is a nontrivial family of deformations, but it is not very large: in the space of all immersions (three functions of two variables), the isometric immersions are generically described by two functions of one variable. Even this result is only valid locally, and we derive our own version under additional assumptions that simplify the situation. To work with the geodesic equation in this space, one wants a manifold structure, which cannot be expected in general: in [33] it has been shown that the space of isometric immersions is in general not locally arcwise connected and hence cannot be a manifold, which gives a counter example to an earlier result of [6]. For the situation of this article we are nevertheless able to overcome these difficulties and show a manifold result, as described below. We suspect that the result is not true in general even for immersions of the flat square.

Our model is a function $\mathbf{r}\colon [0,1]^2 \to \mathbb{R}^3$ satisfying the isometry conditions $|\mathbf{r}_u| = |\mathbf{r}_v| = 1$ and $\mathbf{r}_u \cdot \mathbf{r}_v = 0$, along with the flagpole condition $\mathbf{r}(0,v) = (0,v,0)$. The first task is to classify maps satisfying these conditions. It is well-known that locally such immersions are determined by two functions of one variable; see e.g. the classic text book by do Carmo [11]. With the unit normal given by $N = \mathbf{r}_u \times \mathbf{r}_v$, we define

$$e = \langle \mathbf{r}_{uu}, N\rangle, \quad f = \langle \mathbf{r}_{uv}, N\rangle, \quad \text{and} \quad g = \langle \mathbf{r}_{vv}, N\rangle.$$

The simplifying assumptions

$$e(u,v) > 0 \qquad \text{and} \qquad f(1,v) < 0$$

(corresponding to a furled state of the flag whose geometry is determined by the bottom edge) yield the classification given by Theorem 1, which gives the precise conditions on two functions of one variable needed to generate isometric immersions with our furling conditions. Specifically, under the definitions $\alpha(u) = \int_0^u e(s,0)\,ds$, $\gamma(u) = f(u,0)/e(u,0)$, we find that α and γ must satisfy a list of conditions, and given any functions α and γ satisfying these conditions, we obtain a unique isometric immersion aligned with the flagpole. The proof of this Theorem forms Section 2.

The functions α and γ form a coordinate chart for isometric immersions with our nondegeneracy conditions. Hence we obtain a manifold structure directly by this choice of coordinates; on the other hand we can also prove that the space of isometric immersions satisfying our conditions is a submanifold of the space of all immersions, see Theorem 2. This relies on a precise description of the tangent space and an application of the Implicit Function Theorem for Banach spaces. We do not know if the result can be extended to more general flag states (e.g., when e changes sign across the flag). This tangent space computation and the prrof of the submanifold theorem form Section 3.

Finally, the geodesic equation is obtained from the general principle that geodesics in a submanifold of a flat space satisfy the condition that the acceleration is normal to the submanifold. Deriving this equation relies on computing the orthogonal complement of the tangent space, which turns out to be the most involved computation of this article. This is primarily due to the boundary conditions at the top and bottom edges $v = 0$ and $v = 1$. Ultimately we obtain the geodesic equation of motion given by

$$\mathbf{r}_{tt} = \partial_u(A\mathbf{r}_u) + \partial_v(B\mathbf{r}_u) + \partial_u(B\mathbf{r}_v) + \partial_v(C\mathbf{r}_v) \tag{2}$$

for three functions A, B, C, which act as the "tensions" in a nonlinear wave equation. This comes from the fact that the right side is essentially the second fundamental form of the space of isometric immersions inside the space of all immersions, as we show in Section 4.2. The functions A, B, C are given in terms of the velocity $\mathbf{r}_t$ in a similar way to equation (1): they satisfy the equations

$$\begin{aligned} A_{uu} + B_{uv} - eK &= -\mathbf{r}_{tu} \cdot \mathbf{r}_{tu} \\ A_{uv} + B_{uu} + B_{vv} + C_{uv} - 2fK &= -\mathbf{r}_{tu} \cdot \mathbf{r}_{tv} \\ B_{uv} + C_{vv} - gK &= -\mathbf{r}_{tv} \cdot \mathbf{r}_{tv} \\ K &= eA + 2fB + gC, \end{aligned}$$

together with some rather complicated boundary conditions given in Lemma 6. The rest of Section 4 is devoted to showing how to solve these equations for A, B, and C. The difficulty is that the system is rather degenerate. However, because of this we are able to get quite far with explicit but involved formulas for the solutions; ultimately two rather complicated differential equations for a single function determine A, B, and C, and thus the geodesic equation (2).

1.3. **Future directions.** In future work we plan to continue this line in several directions. First, it would be of particular interest to obtain similar results for a more general class of isometric immersions. The difficulty with actual isometrically embedded surfaces in $\mathbb{R}^3$ is that those without boundary must be fairly rigid, while those with boundary generate very complicated boundary conditions. As a first step to understanding these spaces it might be worth considering surfaces in $\mathbb{R}^4$, as it is much easier to isometrically embed them in this higher dimensional space; for example the tangent space at the standard Clifford torus can be written in terms of functions of two variables, not the single-variable functions that this quasi-rigidity gives us. Hence the theory will be more similar to that for the motion of inextensible closed curves in $\mathbb{R}^2$ or $\mathbb{R}^3$, in terms of computing the tangent space and its orthogonal complement. Although the practical applications are obviously fewer, it would be an interesting space to study and perhaps reveal some information about the geometry of isometric immersions.

Second, from an application oriented point of view we would like to use our geometric framework for the actual modelling of fabric, see for example [14], [9], and [32] and references therein for discussions of current numerical methods for modeling fabric. To discretize the space of isometric immersions (flags resp.) it could be beneficial to use the explicit coordinate chart as obtained in Section 3. Using these coordinates the kinetic energy can also be expressed in terms of two time-dependent functions $\alpha(t, u)$ and $\gamma(t, u)$ via

$$\int_0^T \int_0^1 G_{11}\alpha_t(t,u)^2 + 2G_{12}\alpha_t(t,u)\gamma_t(t,u) + G_{22}\gamma_t(t,u)^2\,du\,dt,$$

where the functions G_{11}, G_{12}, and G_{22} are rather complicated functions depending nonlocally on α and γ. We can then obtain direct equations for α and γ which are solvable directly, and then reconstruct the flag motion afterwards. The drawbacks of this are that the equations are more complicated: while equation (2) looks like a fairly straightforward (nonlinear, nonlocal) wave equation (at least until the boundary conditions enter), in this approach all the coefficients become nonlocal rather quickly. Furthermore we expect equation (2) to be valid generically regardless of the state of the flag (except perhaps under some singular conditions or in a neighborhood of an asymptotic line), while this direct approach will only work as long as our parametrization is valid. Once the flag unfurls, we need to derive new versions of those conditions. On the other hand, the advantage of this approach is that it is obviously easier to discretize the two functions α and γ which satisfy minimal constraints, rather than discretizing the isometry conditions for $\mathbf{r}(u, v)$ directly. Hence we might hope to use numerical methods to simulate the motion of cloth or paper using a minimal set of coordinates, rather than using a high-dimensional system with many constraints (or very strong forces to simulate constraints).

2. The space of flags

We will first introduce the basic notation that we will use throughout the article. In addition we will present some standard results from classic differential geometry for the special case treated in this article, i.e., for isometric immersion of the square in $\mathbb{R}^3$.

In all of this article let $\mathbf{r} \in C^2([0,1]^2, \mathbb{R}^3)$ denote a parametrization of a surface in $\mathbb{R}^3$. Denoting coordinates on $[0,1]^2$ by (u,v) allows us to define the unit normal vector field of the surface:

$$\mathbf{N} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \in C^1([0,1]^2, \mathbb{R}^3)\,.$$

The components of the second fundamental form are then given by

$$e = \mathbf{r}_{uu} \cdot \mathbf{N}, \quad f = \mathbf{r}_{uv} \cdot \mathbf{N}, \quad g = \mathbf{r}_{vv} \cdot \mathbf{N}\,, \tag{3}$$

where $\cdot$ denotes the euclidean vector product in $\mathbb{R}^3$. In this article we are interested in the set of parametrizations that are isometric to euclidean space. Thus the first fundamental form (the metric) is the identity matrix, which is equivalent to the three equations

$$\mathbf{r}_u \cdot \mathbf{r}_u = 1, \quad \mathbf{r}_u \cdot \mathbf{r}_v = 0, \quad \mathbf{r}_v \cdot \mathbf{r}_v = 1\,. \tag{4}$$

Later when analyzing the space of flags, we need the following conditions on the left side of the flag and it's second fundamental form:

$$\mathbf{r}(0,v) = (0,v,0) \tag{5a}$$

$$e(u,v) > 0, \tag{5b}$$

$$f(1,v) < 0. \tag{5c}$$

Remark 1. The last condition (5c) could be replaced by the condition that $f(1,v) > 0$ or that $f(1,v)$ is increasing and then decreasing. This condition is only used later to determine where initial conditions of a certain differential equation can be specified: the current condition implies conditions should be specified at the bottom $v = 0$, while $f(1,v) > 0$ would mean specifying at $v = 1$, and $f(1,v)$ unimodal would involve specifying somewhere in between.

We are now able to define the central object of interest, the space of non-degenerate flags:

Definition 1. The space of *non-degenerate flags* is the set

$$\mathcal{F} = \{\mathbf{r} \in C^\infty([0,1]^2, \mathbb{R}^3) \,|\, \mathbf{r} \text{ satisfies } (4), (5a), (5b), (5c)\}\,.$$

We will also need to consider the space of flags that have only finite regularity, i.e., for $k \geq 2$ we let

$$\mathcal{F}^k = \{\mathbf{r} \in C^k([0,1]^2, \mathbb{R}^3) \,|\, \mathbf{r} \text{ satisfies } (4), (5a), (5b), (5c)\}$$

be the space of flags of regularity C^k.

In the following elementary lemma we present the Gauss-Codazzi equations for flag-type surfaces:

Lemma 1. *Let* $\boldsymbol{r} \in C^k([0,1]^2, \mathbb{R}^3)$ *with* $k \geq 3$*, satisfying equations* (4)*. Then the Gauss-Codazzi equations of* $\boldsymbol{r}$ *are given by*

$$eg = f^2, \quad e_v = f_u, \quad f_v = g_u. \tag{6}$$

Furthermore, the second derivatives of $\boldsymbol{r}$ *satisfy the equations*

$$\boldsymbol{r}_{uu} = e\boldsymbol{N}, \quad \boldsymbol{r}_{uv} = f\boldsymbol{N}, \quad \boldsymbol{r}_{vv} = g\boldsymbol{N}.$$

and the derivatives of N *satisfy*

$$\boldsymbol{N}_u = -e\boldsymbol{r}_u - f\boldsymbol{r}_v,$$
$$\boldsymbol{N}_v = -f\boldsymbol{r}_u - g\boldsymbol{r}_v\,.$$

Note, that the first equation in (6) is simply the statement that the Gauss curvature is zero. These results are standard in classical differential geometry and we refer to [11] for a proof.

2.1. **A characterization of all flag type surfaces.** In the following part we aim to give a complete characterization of all flag type surfaces. Our main result is Theorem 1 that provides a bijection of the space of non-degenerate flags to an open subset of a vector space of functions. A first step towards this result is a characterization of flag surfaces in terms of a boundary condition for the second fundamental form:

Lemma 2. *Let* $\boldsymbol{r} \in \mathcal{F}^k$ *with* $k \geq 3$*. Then the components of the second fundamental form satisfy*

$$f(0,v) = g(0,v) = g_u(0,v) = 0 \quad \text{for } v \in [0,1]. \tag{7}$$

Conversely if these equations and the Gauss-Codazzi-Mainardi equations (6) *are satisfied, then there is an isometric immersion* $\mathbf{r}$ *with second fundamental form having components* (e, f, g) *and satisfying the condition* $\mathbf{r}(0,v) = (0,v,0)$*.*

Proof. If $\mathbf{r}(0,v) = (0,v,0)$, then obviously $g(0,v) = 0$. Since $eg = f^2$, we conclude that $f(0,v) = 0$ as well. Furthermore since $f_v = g_u$, we conclude that $g_u(0,v) = 0$.

Conversely, if $g(0,v) = 0$ we see that $\mathbf{r}_{vv}(0,v) = 0$, so that $v \mapsto \mathbf{r}(0,v)$ is a straight line in $\mathbb{R}^3$; clearly it has length 1 since $|\mathbf{r}_v| = 1$ by equations (4). The Fundamental Theorem of Surfaces says that there is a unique immersion with these components as second fundamental form, up to rotations and translations: if we align this unit-length segment with the flagpole, the immersion is uniquely determined. □

In the following Lemma we will construct a function $\theta(u,v)$, that is both the next step towards a characterization of all flag type surfaces, and will be integral to the L^2-geometry of the space of flags:

Lemma 3. *Let* $k \geq 1$ *and let* e*,* f*,* $g \in C^k([0,1]^2, \mathbb{R})$ *be functions on the square satisfying the system* (6)*. Then there is a function* $\theta \in C^{k+1}([0,1]^2, \mathbb{R})$ *such that*

$$e = \theta_u, \quad f = \theta_v, \quad \theta(0,v) = 0. \tag{8}$$

Proof. Since $e_v = f_u$ by (6), there is a unique function θ such that $\theta_u = e$ and $\theta_v = f$ with $\theta(0,v) = 0$; we simply define $\theta(u,v) = \int_0^u e(s,v)\,ds$ and check that

$$\theta_v(u,v) = \int_0^u f_s(s,v)\,ds = f(u,v) - f(0,v) = f(u,v),$$

using (7). □

The above defined function will be of great importance in the remainder of this article: it will allow us to define a global change of coordinates that will simplify the computations drastically.

Lemma 4. *Let $\boldsymbol{r} \in \mathcal{F}^k$ with $k \geq 3$. Then the functions $x = \theta(u,v)$ and $y = v$ define a global coordinate change with image*

$$\Omega = [0,x^*] \times [0,1] \cup \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leq y \leq 1, x^* \leq x \leq \theta(1,y)\},$$

for some $x^* > 0$. The basic differential operators in the new coordinate system are given by

$$\partial_u = e\partial_x, \qquad \partial_v = f\partial_x + \partial_y$$
$$\partial_x = \frac{1}{e}\partial_u, \qquad \partial_y = -\frac{f}{e}\partial_u + \partial_v$$

Proof. Using (8) the Jacobians of the coordinate changes are

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix}, \quad \frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} \frac{1}{e} & -\frac{f}{e} \\ 0 & 1 \end{bmatrix}$$

Hence $dx \wedge dy = e\,du \wedge dv$, and by the non-degeneracy condition $e > 0$ on $\mathbf{r}$ this is globally invertible. The chain rule yields the change of derivatives. The image is $\Omega = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leq y \leq 1, 0 \leq x \leq \theta(1,y)\}$, which has straight boundaries on the left, top, and bottom, and the graph of some function on the right side, c.f. Figure 1. By condition (5c) the right hand side is a strictly decreasing graph, and there is a value x^* such that Ω splits into the square $[0,x^*] \times [0,1]$ and the domain under the graph. □

The next lemma provides the last missing part for our characterization of all flag type surfaces:

Proposition 1. *Let $\boldsymbol{r} \in \mathcal{F}^k$ with $k \geq 3$. Then there is a function $J \in C^{k-2}([0,L],\mathbb{R})$ with $J(0) = 0$ such that*

$$\theta_v(u,v) = J\big(\theta(u,v)\big)\theta_u(u,v), \qquad \theta(0,v) = 0, \tag{9}$$

and

$$e = \theta_u, \qquad f = \theta_v = J(\theta)e, \qquad g = J(\theta)^2 e = J(\theta)f.$$

Here $L = \sup_{v\in[0,1]} \theta(1,v) = \theta(1,0)$ and $\theta \in C^{k-1}([0,1]^2,\mathbb{R})$ is the function defined by Lemma 3.

Proof. Since e is nowhere zero, we may define a function $\phi = f/e = \theta_v/\theta_u$. Clearly we have $f = \phi e$ and $g = \phi^2 e$ by the Gauss-Codazzi equation (6). To show that J is a function of θ, we change coordinates to (x,y) as in Lemma 4. In these coordinates we have $e\phi_v - f\phi_u = e\phi_y = 0$, so that ϕ is a function of x alone; by the by the non-degeneracy condition we may write

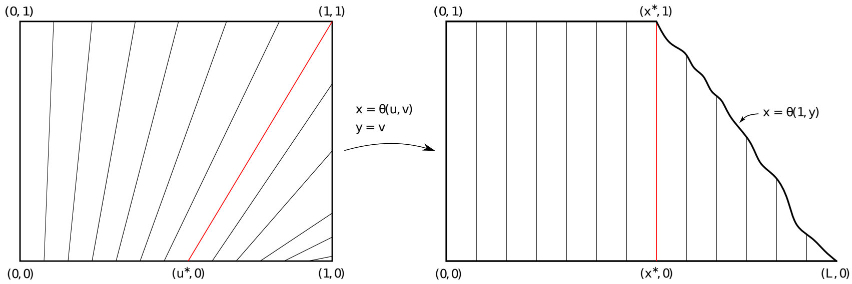

FIGURE 1. Here we demonstrate the image of the transformation from (u, v) coordinates on the square to (x, y) coordinates on a "North Dakota" shape. Because the right side is a decreasing graph, all data can be specified on the bottom edge. We have $x^* = \theta^* = \theta(u^*, 0)$, where u^* is determined by having its characteristic passing through the upper right corner $(1, 1)$. Note that the PDE $\theta_v = J(\theta)\theta_v$ has all characteristics consisting of straight lines which become vertical in the new coordinates.

$\phi = J(x)$ where $J\colon [0, L] \to \mathbb{R}$, with $L = \sup_{v\in[0,1]} \theta(1, v)$. Changing back into (u, v) coordinates yields $\phi = J \circ \theta$. The fact that $J(0) = 0$ comes from the boundary condition (7). □

Remark 2 (The condition (5c))**.** Notice here the importance of the condition on f on the right side of the flag. If this condition is not satisfied, the equation $\phi_y = 0$ could not be solved by $\phi(x) = J(x)$ as the characteristics $(x, y) = (x_0, t)$ would leave and reenter the domain.

We are now able to formulate our main result of this section, which characterizes all non-degenerate flags in terms of two functions:

Theorem 1. *Let $\boldsymbol{r} \in \mathcal{F}^k$ with $k \geq 3$ and let $\theta \in C^{k-1}([0, 1]^2, \mathbb{R})$ and $J \in C^{k-2}([0, L], \mathbb{R})$ be the corresponding functions from Lemma 3 and Proposition 1.*

Let $\alpha \in C^{k-1}([0, 1], \mathbb{R})$ be the function defined by $\alpha(u) = \theta(u, 0)$, and let $\gamma \in C^{k-2}([0, 1], \mathbb{R})$ be the function defined by $\gamma(u) = J(\alpha(u))$. Then α and γ satisfy the constraints

- *$\alpha(0) = 0$ and $\alpha'(u) > 0$ for $u \in [0, 1]$.*
- *$\gamma(0) = 0$ and $\gamma(u^*) = u^* - 1$ for some unique $u^* \in (0, 1)$.*
- *$\gamma'(u) < 1$ for $u \in [0, u^*]$.*
- *$\frac{d}{du}\frac{\gamma(u)}{1-u} < 0$ for $u \in [u^*, 1]$.*

Conversely for any $k \geq 3$ and pair of functions $\alpha \in C^{k-1}([0, 1], \mathbb{R})$ and $\gamma \in C^{k-2}([0, 1], \mathbb{R})$ satisfying these four conditions, there exist a flag $\mathbf{r} \in \mathcal{F}^k$ with $J = \gamma \circ \alpha^{-1}$ and $\theta(u, 0) = \alpha(u)$.

Remark 3 (A chart for $\mathcal{F}^k$)**.** The main observation here is that the functions α and γ, with the conditions stated, generate any flag. Thus this result can be interpreted as providing a coordinate chart for the manifold of non-degenerate flags. We will however show a stronger (sub)manifold result using the inverse function theorem in Section 3.

Proof. Suppose θ is given. The first statement is easy: since $\alpha(0) = \theta(0,0)$ and $\alpha'(u) = \theta_u(u,0) = e(u,0)$, the conditions (9) and (5b) immediately yield that $\alpha(0) = 0$ and $\alpha'(u) > 0$. Since α is invertible, we may define $z\colon [0,1]^2 \to \mathbb{R}$ by $\theta = \alpha \circ z$; then since $\gamma = J \circ \alpha$, equation (9) implies that z satisfies the PDE

$$z_v = (\gamma \circ z) z_u, \qquad z(u,0) = u.$$

It will be more convenient to work with z. Note first that our assumptions (5b) and (5c) imply that $z_u(u,v) > 0$ and $z_v(1,v) < 0$. Since $\sup \theta(u,v) = \theta(1,0) = L$, we see that $\sup z(u,v) = z(1,0) = 1$, and we conclude that $0 < z(1,1) < 1$.

We may easily check that the function $H(u,v) = v\gamma\big(z(u,v)\big) + u - z(u,v)$ satisfies $H_v = (\gamma \circ z) H_u$ as well as $H(u,0) = 0$ for $0 \le u \le 1$; the unique solution of this linear PDE is $H(u,v) \equiv 0$ in a neighborhood of the bottom line $v = 0$. We obtain

$$v\gamma\big(z(u,v)\big) + u = z(u,v). \tag{10}$$

In particular when $u = v = 1$ then $\gamma(x^*) + 1 = x^*$, so that $x^* = z(1,1)$.

Thus equation (10) with $H(u,v) = 0$ gives an algebraic equation for z, and differentiating this equation yields

$$v\gamma'(z) z_u + 1 = z_u, \qquad \gamma(z) + v\gamma'(z) z_v = z_v,$$

and solving for z_u and z_v gives

$$z_u = \frac{1}{1 - v\gamma'\big(z(u,v)\big)}, \qquad z_v = \frac{\gamma\big(z(u,v)\big)}{1 - v\gamma'\big(z(u,v)\big)}. \tag{11}$$

Since $z_u > 0$ we must have $1 - v\gamma'\big(z(u,v)\big) > 0$ for all (u,v), and in particular $1 - \gamma'(z(u,1)) > 0$. Since $0 \le z(u,1) \le x^*$ we conclude that $\gamma'(t) < 1$ for $0 \le t \le x^*$.

On the other hand $z_v(1,v) < 0$ implies that the function $v \mapsto \frac{z(1,v)}{1 - v\gamma'(z(1,v))}$ is negative. Using equation (10) we may solve for v in terms of $z(1,v)$ to get $v = \frac{z(1,v)-1}{\gamma(z(1,v))}$. Letting $t = z(1,v)$ which increases from x^* to 1 as v decreases from 1 to 0, we obtain $z_v(1,v) = \frac{\gamma(t)}{1 - \frac{(t-1)\gamma'(t)}{\gamma(t)}}$. The condition that this is negative is easily seen to be equivalent to $\frac{d}{dt}\frac{\gamma(t)}{1-t} < 0$.

Conversely given γ satisfying the itemized conditions above, we can check that the function

$$F(z) = v\gamma(z) + u - z$$

satisfies, for each fixed (u,v), the conditions $F(0) = u \ge 0$ and $F(1) = v\gamma(1) + u - 1$. Since $\gamma(1) < 0$ by our assumptions, we have $F(1) \ge 0$. Furthermore $F'(z) = v\gamma'(z) - 1$; for $z \le x^*$ we have $F'(z) < v - 1 \le 0$, while for $z \ge x^*$ we have $F'(z) < v\frac{\gamma(z)}{1-z} - 1 < 0$, and thus F is strictly decreasing. There is thus a unique solution of $F(z) = 0$ for each (u,v), and we obtain a function $z(u,v)$ which is continuously differentiable by the implicit function theorem. The calculation (11) shows that z satisfies the partial differential equation $z_v = (\gamma \circ z) z_u$, and thus $\theta = \alpha \circ z$ satisfies the PDE $\theta_v = (J \circ \theta)\theta_u$. Finally the previous paragraph shows that $z_u(u,v) > 0$ and $z_v(1,v) < 0$, which in turn imply the formulas (5b) and (5c), as desired.

□

3. The manifold structure of the space of flags

In this section we will prove that the space of flags can be viewed as a submanifold of the space of all C^k-maps:

Theorem 2. *For any $k \geq 2$ the space of non-degenerate flags $\mathcal{F}^k$ is a Banach submanifold of the space $C^k([0,1]^2, \mathbb{R}^3)$. The tangent space at a flag $\mathbf{r}$ is defined by the system of PDEs*

$$a_u - ec = 0, \quad a_v + b_u - 2fc = 0, \quad b_c - gc = 0. \tag{12}$$

Here e, f, g are the components of the second fundamental form of $\mathbf{r}$ and a, b, c are the components of the tangent vector $\mathbf{h} = a\mathbf{r}_u + b\mathbf{r}_v + c\mathbf{N}$, with boundary conditions given by

$$a(0,v) = b(0,v) = c(0,v) = 0. \tag{13}$$

Remark 4 (Manifold structure of general spaces of isometric immersions)**.** We want to remark that these results are non-trivial and evenmore, are not true in the context of the space of general isometric immersions: in the article [33] it was shown that the space of isometric immersions of S^2 into $\mathbb{R}^3$ is not locally arcwise connected and thus cannot be a manifold. This example disproved the earlier result of [6]. To our knowledge there have been no new results in this direction since then and this is the first non-trivial example of a space of isometric immersions that can be equipped with a (sub)manifold structure.

We aim to apply the inverse function theorem to show this result. Therefore we define the map

$$\Phi : \begin{cases} C^k([0,1]^2, \mathbb{R}^3) & \to C^{k-1}([0,1]^2, \mathbb{R}^3) \\ \mathbf{r} & \mapsto \left(\frac{1}{2}\mathbf{r}_u \cdot \mathbf{r}_u, \mathbf{r}_u \cdot \mathbf{r}_v, \frac{1}{2}\mathbf{r}_v \cdot \mathbf{r}_v\right). \end{cases}$$

Note that $\Phi(\mathcal{F}^k) = (1/2, 0, 1/2)$. In the following we will show that the map Φ is sufficiently well-behaved, which will allow us to obtain our submanifold result using the inverse function theorem. Therefore we define for $k \geq 2$ the affine space of non-degenerate C^k-mappings satisfying the flag pole condition:

$$C^k_*([0,1]^2, \mathbb{R}^3) := \Big\{ \mathbf{r} \in C^k([0,1]^2, \mathbb{R}^3) : \\ \mathbf{r}(0,v) = (0,v,0) \text{ and } \mathbf{r} \text{ satisfies (5b), (5c)} \Big\}.$$

Note that $C^k_*([0,1]^2, \mathbb{R}^3)$ is an open supset of an affine space and thus it is a Banach manifold itself. We have

Lemma 5. *Let $k \geq 3$. Then the map*

$$\Phi : C^k_*([0,1]^2, \mathbb{R}^3) \to C^{k-1}([0,1]^2, \mathbb{R}^3)$$

is a C^{k-1}-map. For each $\mathbf{r} \in \mathcal{F}^k \subset C^k_*([0,1]^2, \mathbb{R}^3)$ the tangent mapping

$$d\Phi_{\mathbf{r}} : T_{\mathbf{r}} C^k_*([0,1]^2, \mathbb{R}^3) \to \Big\{ (f^1, f^2, f^3) \in C^{k-1}([0,1]^2, \mathbb{R}^3) : f^3(0,v) = 0 \Big\}$$

is surjective.

Proof. The differentiability of the map is clear. It remains to show that

$$d\Phi_{\mathbf{r}}(\mathbf{h}) = \mathbf{f} \tag{14}$$

has a solution

$$\mathbf{h} \in T_{\mathbf{r}} C^k_*([0,1]^2, \mathbb{R}^3) = \Big\{ \mathbf{h} \in C^k([0,1]^2, \mathbb{R}^3) : \mathbf{h}(0,v) = (0,0,0) \Big\}$$

for any $\mathbf{r} \in \mathcal{F}^k$ and $\mathbf{f} \in C^{k-1}([0,1]^2, \mathbb{R}^3)$ with $f^3(0,v) = 0$. We first calculate the derivative of the map Φ. We have:

$$d\Phi_r(h) = (\mathbf{h}_u \cdot \mathbf{r}_u, \mathbf{h}_u \cdot \mathbf{r}_v + \mathbf{h}_v \cdot \mathbf{r}_u, \mathbf{h}_v \cdot \mathbf{r}_v)\,. \tag{15}$$

Now we decompose the variation vector $\mathbf{h}$ in its components via

$$\mathbf{h} = a\mathbf{r}_u + b\mathbf{r}_v + c\mathbf{N}.$$

and write $\mathbf{f}(u,v) = (f^1(u,v), f^2(u,v), f^3(u,v))$. Using Lemma 1, the derivatives of $\mathbf{h}$ can be written as

$$\begin{aligned} \mathbf{h}_u &= (a_u - ec)\mathbf{r}_u + (b_u - fc)\mathbf{r}_v + (c_u + ea + fb)\mathbf{N} \\ \mathbf{h}_v &= (a_v - fc)\mathbf{r}_u + (b_v - gc)\mathbf{r}_v + (c_v + fa + gb)\mathbf{N} \end{aligned}$$

Plugging this into (15) yields the system of PDEs

$$a_u - ec = f^1, \quad a_v + b_u - 2fc = f^2, \quad b_v - gc = f^3. \tag{16}$$

The boundary conditions can be deduced by writing $\mathbf{h} = \mathbf{r}_t(0,u,v)$ where $\mathbf{r}(t,u,v)$ is a path of flags satisfying $\mathbf{r}(t,0,v) = (0,v,0)$, hence at $t = 0$ we must have

$$a(0,v) = b(0,v) = c(0,v) = 0. \tag{17}$$

This imposes the condition $f^3(0,v) = 0$.

In order to solve this system we switch to $(x,y) = \big(\theta(u,v), v\big)$ coordinates as defined in Lemma 4. In these coordinates the equations take the form

$$a_x - c = f^1, \qquad b_y + fb_x - gc = f^2, \qquad fa_x + a_y + eb_x - 2cf = f^3.$$

The first equation determines c in terms of a and f^1, leaving us with

$$b_y + fb_x - ga_x = f^2 - gf^1 \tag{18}$$

$$a_y + eb_x - a_x f = f^3 - 2ff^1 \tag{19}$$

Using the fact that $f = Je$ and $g = Jf$, with J depending on x only, we subtract J times (19) from (18) and find

$$\frac{d}{dy}(b_y - J(x)a) = f^2 - Jf^3 + gf^1 := \tilde{f}.$$

We integrate to find

$$b(x,y) = J(x)a(x,y) + \zeta(x) + \int_0^y \tilde{f}(x,\tau)\,d\tau$$

for some function $\zeta(x)$. Note that b satisfies the correct boundary condition. Plugging this into (19), we obtain the equation

$$a_y + e(J'a + Ja_x + \zeta' + \int_0^y \tilde{f}_x(x,\tau)\,d\tau) - fa_x = 0$$

Which simplifies since $f = Je$, to

$$a_y(x,y) + e(x,y)J'(x)a(x,y) + e(x,y)\zeta'(x) + e\int_0^y \tilde{f}_x(x,\tau)\,d\tau = 0.$$

The dependence of e on x and y is somewhat simple; using (11) and the fact that $\theta = \alpha(z)$ we have

$$e = \theta_u = \frac{\alpha'(z)}{1-\gamma'(z)v} = \frac{1}{\beta'(x) - J'(x)y}. \tag{20}$$

This yields

$$\frac{\partial a}{\partial y} + \frac{J'(x)}{\beta'(x) - J'(x)y}a = -\frac{\zeta'(x) + \int_0^y \tilde{f}_x(x,\tau)\,d\tau}{\beta'(x) - J'(x)y},$$

which is an ODE in y. This can be solved explicitly as

$$a(x,y) = (a_0(x)(\beta'(x) - J'(x)y)) - \frac{\zeta'(x) + \int_0^y \tilde{f}_x(x,\tau)\,d\tau}{J'(x)}$$

for some function $a_0(x)$. Any choice of $a_0(x)$ and $\zeta(x)$ will yield a solution of the equation (14), which concludes the proof. □

Proof of Theorem. Using Lemma 5 the submanifold result follows by the regular value theorem. The equation for the tangent space is simply

$$d\Phi_{\mathbf{r}}(\mathbf{h}) = (0,0,0)\ ,$$

which yields to the explicit description by setting $\mathbf{f} = (0,0,0)$ in (16). □

Remark 5. We decided to present all results in terms of C^k-immersions. Using a similar argument one would obtain the result for other function spaces, e.g., isometric immersions of Sobolev class H^k for k sufficiently large.

In the following result we present an explicit solution to the tangent space equations, which will be important for the derivation of the geodesic equation in Section 4:

Corollary 1. *For a non-degenerate flag* $\mathbf{r}$*, the equations* (12) *with boundary conditions* (13) *have the solutions*

$$\begin{aligned} a(u,v) &= p(\theta(u,v)) + vs(\theta(u,v)),\\ b(u,v) &= q(\theta(u,v)) + vJ(\theta(u,v))s(\theta(u,v))\\ c(u,v) &= p'(\theta(u,v)) + vs'(\theta(u,v)) \end{aligned} \tag{21}$$

where $s(x)$ *is given in terms of* $p(x)$ *and* $q(x)$ *by*

$$\beta'(x)s(x) = J(x)p'(x) - q'(x),$$

and α*,* γ *are functions given in Theorem 1,* β *is the inverse of* α*, and* p *and* q *are some functions satisfying*

$$p(0) = 0, \qquad p'(0) = 0, \qquad q(0) = 0, \qquad q'(0) = 0, \qquad q''(0) - 0. \tag{22}$$

Proof. Continue as in the proof of Lemma 5, and set $\mathbf{f} = (0,0,0)$, which implies that $\tilde{f} = 0$. Writing $p(x) = a_0(x)\beta'(x) - \zeta'(x)/J'(x)$, $s(x) = -a_0(x)J'(x)$, and $q(x) = \zeta(x) + J(x)a_0(x)\beta'(x) - \zeta'(x)J(x)/J'(x)$, we see that a and b are given by (21) with $J(x)p'(x) - q'(x) = \beta'(x)s(x)$. Finally the formula for c comes from $c = a_x$.

The boundary conditions (22) come from the conditions (17), which must be satisfied at $u = 0$ for all v, using the fact that $J(0) = 0$ and $\beta'(0) \neq 0$. $\square$

4. The motion of a flag

In this section we define the L^2 metric on the space of non-degenerate flags and derive its formal geodesic equation. These equations are the governing equation for the motion of a flag without gravity, as the L^2-metric can be interpreted as the kinetic energy of a flag.

Remark 6. The reason we call the geodesic equation formal is the loss of regularity that appears in the geodesic spray, cf. [12, 7]. Thus the spray is not a vector field on $T\mathcal{F}^k$, and the geodesic equation does not exist in the C^k-category, but can only be interpreted in an appropriate weak sense in a bigger space. In the smooth category $\mathcal{F}$ this is not an issue. However we do not discuss the submanifold result for the smooth category in this article, which would require the use of the Nash-Moser inverse function theorem.

4.1. **The L^2-metric.** For any functions $\mathbf{h}, \mathbf{k} \in C^k([0,1]^2, \mathbb{R}^3)$ we consider the L^2-product given by

$$\langle \mathbf{h}, \mathbf{k} \rangle_{L^2} = \int_0^1 \int_0^1 \langle \mathbf{h}(u,v), \mathbf{k}(u,v) \rangle_{\mathbb{R}^3} \, du \, dv \, .$$

On the space of all C^k-mappings this is a flat Riemannian metric and geodesics are simply given by straight lines, see e.g. [12, 25].

In the following we will consider the restriction of this Riemannian metric to the space of non-degenerate flags $\mathcal{F}^k \subset C^k([0,1]^2, \mathbb{R}^3)$. Restricted to the submanifold $\mathcal{F}^k$, the geodesics will be non-trivial and to calculate them one has to solve a non-linear PDE, called the geodesic equation. It is the aim of the remainder of this article to derive these equations, which govern the motion of flags. As mentioned above, the L^2-product can be interpreted as the kinetic energy of a flag and thus these equations can be used to model the motion of a flag without gravity.

4.2. **The Orthogonal Projection.** The geodesic equation of a submanifold of a flat space is intimately linked to the orthogonal complement of the tangent space. As a first result we will prove that a certain class of vector fields is an element of this complement. Inspired by the formula $\partial_u(A\mathbf{r}_u)$ for the orthogonal component of a vector in the case of inextensible threads, c.f. [28, 29], we guess that vectors of the form

$$\mathbf{w} = \partial_u(A\mathbf{r}_u + B\mathbf{r}_v) + \partial_v(B\mathbf{r}_u + C\mathbf{r}_v). \tag{23}$$

are in the orthogonal complement. I turns out, that it is easy to see that any such field is orthogonal to any tangent vector after imposing the correct boundary conditions:

Lemma 6. *Any field z tangent to the space of flags, with components satisfying (12) and (13), is L^2-orthogonal to any field w of the form (23), as long as the components of w satisfy the following boundary conditions:*

$$\begin{aligned} \eta'(x)^2 J(x)\big(A(x,\eta(x)) + J(x)B(x,\eta(x))\big) \\ + \beta'(x)^2\big(B(x,0) + J(x)C(x,0)\big) = 0 \end{aligned} \tag{24}$$

$$\frac{d}{dx}\Big(\frac{\eta(x)\eta'(x)}{\beta'(x)}\big[A(x,\eta(x))+J(x)B(x,\eta(x))\big]\Big) + \frac{\beta'(x)-\eta(x)J'(x)}{J(x)}B(x,\eta(x))+\beta'(x)C(x,0)=0 \tag{25}$$

on the interval $x\in[\theta^*,L]$*, and*

$$\big[\beta'(x)-J'(x)\big]^2\big[B(x,1)+J(x)C(x,1)\big] - \beta'(x)^2\big[B(x,0)+J(x)C(x,0)\big]=0 \tag{26}$$

$$\frac{d}{dx}\Big(\frac{\beta'(x)-J'(x)}{\beta'(x)}\big[B(x,1)+J(x)C(x,1)\big]\Big) + [\beta'(x)-J'(x)]C(x,1)-\beta'(x)C(x,0)=0 \tag{27}$$

on the interval $x\in[0,\theta^*]$*, where* $\beta=\alpha^{-1}$ *and* $\eta(x)=\frac{\beta(x)-1}{J(x)}$ *describes the right side of the flag via* $y=\eta(x)$*. In addition at the point* θ^* *we must have*

$$e(\theta^*,1)A(\theta^*,1)+2f(\theta^*,1)B(\theta^*,1)+g(\theta^*,1)C(\theta^*,1)=0\,.$$

Proof. We decompose the field tangent to the space of isometric immersion in its normal and tangent components $\mathbf{z}=a\mathbf{r}_u+b\mathbf{r}_v+c\mathbf{N}$ as in Theorem 2. We then change variables to $x=\theta(u,v)$ and $y=v$ and recall, c.f. Lemma 4, that the image of the square in (x,y) space consists of the rectangle $[0,\theta^*]\times[0,1]$ together with the region under the graph of the right side; see Figure 1. Furthermore, the basic differential operators in $(u,v)-$coordinates are given by $\frac{\partial}{\partial u}=e\,\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial v}=f\,\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$. In these variables the conditions for z to be tangent to the space of isometric immersions translate to

$$e\mathbf{z}_x\cdot\mathbf{r}_u=0,\qquad e\mathbf{z}_x\cdot\mathbf{r}_v+(f\mathbf{z}_x+\mathbf{z}_y)\cdot\mathbf{r}_u=0,\qquad (f\mathbf{z}_x+\mathbf{z}_y)\cdot\mathbf{r}_v=0. \tag{28}$$

Here we have just rewritten $d\Phi_{\mathbf{r}}(\mathbf{h})=0$, c.f. equation (15), to the new coordinate system. The change of variables transformation has Jacobian determinant given by $dx\wedge dy=e(x,y)\,du\wedge dv$, and by formula (20) we get $du\,dv=\big[\beta'(x)-yJ'(x)\big]\,dx\,dy$.

Note that the right side of the flag is parameterized by $x=\alpha(z(1,v))$ and $y=v$, where $z(1,v)-v\gamma(z(1,v))=1$. Now $z(1,v)=\beta(x)$, so that $\beta(x)-y\gamma(\beta(x))=1$. We obtain $y=\frac{\beta(x)-1}{\gamma(\beta(x))}$, which is $y=\eta(x)$ in our notation since $J(x)=\gamma(\beta(x))$. Now let

$$\xi=\eta^{-1};$$

the condition that z be orthogonal to w takes the form, in (x,y) coordinates, that

$$\int_0^1\int_0^{\xi(y)}\Big(\mathbf{z}(x,y)\cdot\partial_x(A\mathbf{r}_u+B\mathbf{r}_v)+J(x)z(x,y)\cdot\partial_x(B\mathbf{r}_u+C\mathbf{r}_v) + \big[\beta'(x)-yJ'(x)\big]z(x,y)\cdot\partial_y(B\mathbf{r}_u+C\mathbf{r}_v)\Big)\,dx\,dy=0.$$

Using (28) integration by parts shows that all interior terms vanish. Thus we are left with the boundary terms, which need to be handled carefully.

Changing the order of integration as needed to facilitate integrating by parts, keeping in mind Figure 1, we obtain:

$$\int_0^1 \mathbf{z}\cdot(A\mathbf{r}_u + B\mathbf{r}_v)\Big|_{x=0}^{x=\xi(y)} + J\mathbf{z}\cdot(B\mathbf{r}_u + C\mathbf{r}_v)\,dy \\ + \int_0^{\theta^*} (\beta'(x) - yJ'(x))\mathbf{z}\cdot(B\mathbf{r}_u + C\mathbf{r}_v)\Big|_{y=0}^{y=1} dx \\ + \int_{\theta^*}^{L} (\beta'(x) - yJ'(x))\mathbf{z}\cdot(B\mathbf{r}_u + C\mathbf{r}_v)\Big|_{y=0}^{y=\eta(x)} dx = 0.$$

Decomposing $\mathbf{z} = a\mathbf{r}_u + b\mathbf{r}_v + c\mathbf{N}$ we then obtain

$$\int_0^1 \Big((aA+bB)+J(aB+bC)\Big)\Big|_{x=\xi(y)} dy + \int_0^{\theta^*} [\beta'(x)-yJ'(x)](aB+bC)\Big|_{y=0}^{y=1} dx \\ + \int_{\theta^*}^{L} [\beta'(x) - yJ'(x)](aB+bC)\Big|_{y=0}^{y=\eta(x)} dx = 0.$$

Here we use that for $x = 0$ we have $a = b = 0$. In the first term we change variables via $y = \eta(x)$, so the limits go from L to θ^*; we obtain

$$\begin{aligned}
(29)\qquad & -\int_{\theta^*}^{L} \big(a(x,\eta(x))\big[A(x,\eta(x)) + J(x)B(x,\eta(x))\big]\eta'(x)\,dx \\
& - \int_{\theta^*}^{L} b(x,\eta(x))\big[B(x,\eta(x)) + J(x)C(x,\eta(x))\big]\eta'(x)\,dx \\
& + \int_0^{\theta^*} \big[\beta'(x) - J'(x)\big]\big[a(x,1)B(x,1) + b(x,1)C(x,1)\big]\,dx \\
& - \int_0^{\theta^*} \beta'(x)\big[a(x,0)B(x,0) + b(x,0)C(x,0)\big]\,dx \\
& + \int_{\theta^*}^{L} \big[\beta'(x) - \eta(x)J'(x)\big]\big[a(x,\eta(x))B(x,\eta(x)) + b(x,\eta(x))C(x,\eta(x))\big]\,dx \\
& - \int_{\theta^*}^{L} \beta'(x)\big[a(x,0)B(x,0) + b(x,0)C(x,0)\big]\,dx = 0.
\end{aligned}$$

Now since $\eta(x) = \frac{\beta(x)-1}{J(x)}$, we see that $\beta'(x) - \eta'(x)J(x) - \eta(x)J'(x) = 0$. Thus the condition (29) simplifies slightly to

$$\begin{aligned}
(30)\qquad & \\
& -\int_{\theta^*}^{L} \big(a(x,\eta(x))A(x,\eta(x))\eta'(x)\,dx - \int_{\theta^*}^{L} b(x,\eta(x))B(x,\eta(x))\eta'(x)\,dx \\
& + \int_0^{\theta^*} \big[\beta'(x) - J'(x)\big]\big[a(x,1)B(x,1) + b(x,1)C(x,1)\big]\,dx \\
& - \int_0^{\theta^*} \beta'(x)\big[a(x,0)B(x,0) + b(x,0)C(x,0)\big]\,dx \\
& - \int_{\theta^*}^{L} \beta'(x)\big[a(x,0)B(x,0) + b(x,0)C(x,0)\big]\,dx = 0.
\end{aligned}$$

Using $\eta'(x) = [\beta'(x) - \eta(x)J'(x)]/J(x)$ and writing $\tilde{A}(x,y) = [\beta'(x) - yJ'(x)]A(x,y)$ (with similar formulas for $\tilde{B}$ and $\tilde{C}$), this becomes

$$-\int_{\theta^*}^{L} a(x,\eta(x))\tilde{A}(x,\eta(x))/J(x)\,dx - \int_{\theta^*}^{L} b(x,\eta(x))\tilde{B}(x,\eta(x))/J(x)\,dx \\ + \int_0^{\theta^*} \big[a(x,1)\tilde{B}(x,1) + b(x,1)\tilde{C}(x,1)\big]\,dx \\ - \int_0^{L} \big[a(x,0)\tilde{B}(x,0) + b(x,0)\tilde{C}(x,0)\big]\,dx.$$

We may now plug in our formulas for a and b from (21): in (x,y) coordinates they are

$$a(x,y) = p(x) + ys(x) \qquad \text{and} \qquad b(x,y) = q(x) + yJ(x)s(x),$$

where

$$\beta'(x)s(x) = J(x)p'(x) - q'(x). \tag{31}$$

Collecting terms that depend on p, q, and s separately, (30) becomes

$$\int_0^{\theta^*} p(x)\big(\tilde{B}(x,1) - \tilde{B}(x,0)\big)\,dx \\ + \int_0^{\theta^*} q(x)\big(\tilde{C}(x,1) - \tilde{C}(x,0)\big)\,dx \\ + \int_0^{\theta^*} s(x)\Big(\tilde{B}(x,1) + J(x)\tilde{C}(x,1)\Big)\,dx \\ + \int_{\theta^*}^{L} p(x)\Big(-\tilde{A}(x,\eta(x))/J(x) - \tilde{B}(x,0)\Big)\,dx \\ + \int_{\theta^*}^{L} q(x)\Big(-\tilde{B}(x,\eta(x))/J(x) - \tilde{C}(x,0)\Big)\,dx \\ + \int_{\theta^*}^{L} s(x)\Big(-\eta(x)\tilde{A}(x,\eta(x))/J(x) - \eta(x)\tilde{B}(x,\eta(x))\Big)\,dx = 0.$$

Finally we use (31) to integrate by parts and eliminate s. Letting

$$\phi(x) = \frac{\eta(x)[\tilde{A}(x,\eta(x)) + J(x)\tilde{B}(x,\eta(x))}{J(x)\beta'(x)} \qquad \text{and} \tag{32}$$

$$\psi(x) = \frac{\tilde{B}(x,1) + J(x)\tilde{C}(x,1)}{\beta'(x)}, \tag{33}$$

we obtain

$$\int_0^{\theta^*} p(x)\Big(\tilde{B}(x,1)-\tilde{B}(x,0)-\frac{d}{dx}\big(J(x)\psi(x)\big)\Big)\,dx \\ +\int_0^{\theta^*} q(x)\Big(\tilde{C}(x,1)-\tilde{C}(x,0)+\psi'(x)\Big)\,dx \\ +\int_{\theta^*}^{L} p(x)\Big(-\tilde{A}(x,\eta(x))/J(x)-\tilde{B}(x,0)+\frac{d}{dx}\big(J(x)\phi(x)\big)\Big)\,dx \\ +\int_{\theta^*}^{L} q(x)\Big(-\tilde{B}(x,\eta(x))/J(x)-\tilde{C}(x,0)-\phi'(x)\Big)\,dx \\ +\big[p(\theta^*)-q(\theta^*)/J(\theta^*)\big]\,\frac{\tilde{P}(\theta^*,1)+J(\theta^*)\tilde{Q}(\theta^*,1)}{\beta'(\theta^*)}=0.$$

In this equation we used the fact that $\eta(L)=0$ to cancel out the far-right boundary term.

Now p and q are arbitrary functions of the variable x (on both subintervals $[0,\theta^*]$ and $[\theta^*,L]$), and for this quantity to vanish for any choice of p and q, we must have

$$\begin{aligned}\tilde{B}(x,1)-\tilde{B}(x,0)-J(x)\psi'(x)-J'(x)\psi(x)&=0\\ \tilde{C}(x,1)-\tilde{C}(x,0)+\psi'(x)&=0\end{aligned} \tag{34}$$

on the subinterval $[0,\theta^*]$, and

$$\begin{aligned}-\tilde{A}(x,\eta(x))/J(x)-\tilde{B}(x,0)+J'(x)\phi(x)+J(x)\phi'(x)&=0\\ -\tilde{B}(x,\eta(x))/J(x)-\tilde{C}(x,0)-\phi'(x)&=0\end{aligned} \tag{35}$$

on the subinterval $[\theta^*,L]$. In addition since $p(\theta^*)$ and $q(\theta^*)$ are arbitrary numbers, we must have

$$\tilde{P}(\theta^*,1)+J(\theta^*)\tilde{Q}(\theta^*,1)=K(\theta^*,1)/e(\theta^*,1)=0,$$

so that

$$K(\theta^*,1)=0.$$

Using the definition (33) of $\psi(x)$ in (34), then eliminating $\psi'(x)$ and simplifying, we obtain the condition (26), while the second equation of (34) is already (27). Similarly using the definition (32) of $\phi(x)$ in (35), then eliminating $\phi'(x)$ and simplifying, we obtain (24), while the second equation is already (25). □

We could now proceed with deriving a general formula for the orthogonal projection. However, this is more complicated than necessary since we only need the orthogonal projection of certain tangent vectors to calculate the geodesic equation. To make the presentation not more technical than necessary we will thus proceed without presenting the general case.

4.3. **The geodesic equation.** To calculate the geodesic equation we will need to make use of the submanifold structure of the space of flags $\mathcal{F}^k$ as a subset of the space C^k. The geodesic equation is then given by:

$$\nabla^{\mathcal{F}^k}_{\mathbf{r}_t}\mathbf{r}_t=\mathbf{r}_{tt}-S^{\mathcal{F}^k}(\mathbf{r}_t,\mathbf{r}_t)=0\,,$$

where $S^{\mathcal{F}^k}: T_{\mathbf{r}}\mathcal{F}^k \times T_{\mathbf{r}}\mathcal{F}^k \to \left(T_{\mathbf{r}}\mathcal{F}^k\right)^{\perp}$ is the second fundamental form of the submanifold of all flags as a subset of the space of C^k-functions. Here we used that the covariant derivative of the L^2-metric on $C^k([0,1]^2, \mathbb{R}^3)$ is given by $\nabla^{C^k}_{\mathbf{r}_t}\mathbf{r}_t = \mathbf{r}_{tt}$. To calculate the second fundamental form we will be following the method of [29], see also [5]. Therefore we let U and V be vector fields on $\mathcal{F}^k$ with value u and v when evaluated at $\mathbf{r}_0$. Then the second fundamental form is given as the orthogonal projection of the covariant derivative of V in direction U evaluated at $\mathbf{r}_0$, i.e.,

$$S^{\mathcal{F}^k}(u,v) = \left(\nabla^{C^k}_U V\right)^{\perp}_{\mathbf{r}_0},$$

where where $(\cdot)^{\perp}$ denotes the orthogonal projection with respect to the L^2-metric For the purpose of deriving the geodesic equation we are only interested in the second fundamental form of the specific choice $u = v = \mathbf{r}_t$, i.e., we need to calculate the orthogonal projection of $\mathbf{r}_{tt}$. Therefore we will first derive a new set of equations for the orthogonal part of a vector that is the second derivative of a path of isometric immersions:

Lemma 7. *Suppose $t \mapsto \mathbf{r}(t,u,v)$ is a curve in the space of flags, where the components of $\mathbf{r}_t$ satisfy the conditions (21). Suppose that $\mathbf{r}_{tt} = z + w$, where z satisfies the tangent conditions (12) and w takes the form (23) with components A, B, and C satisfying (24)–(27). If we write*

$$H = A_u + B_v, \qquad I = B_u + C_v, \qquad K = eA + 2fB + gC, \tag{36}$$

then H, I, and K satisfy the system

$$\begin{aligned} H_u - eK &= -|\mathbf{r}_{tu}|^2, \\ H_v + I_u - 2fK &= -2\mathbf{r}_{tu}\cdot\mathbf{r}_{tv}, \\ I_v - gK &= -|\mathbf{r}_{tv}|^2. \end{aligned} \tag{37}$$

Proof. The proof is straightforward. Since z satisfies the tangent conditions, we must have

$$\begin{aligned} \langle w_u, \mathbf{r}_u\rangle &= \langle \mathbf{r}_{ttu}, \mathbf{r}_u\rangle \\ \langle w_u, \mathbf{r}_v\rangle + \langle w_v, \mathbf{r}_u\rangle &= \langle \mathbf{r}_{ttu}, \mathbf{r}_v\rangle + \langle \mathbf{r}_{ttv}, \mathbf{r}_u\rangle \\ \langle w_v, \mathbf{r}_v\rangle &= \langle \mathbf{r}_{ttv}, \mathbf{r}_v\rangle. \end{aligned} \tag{38}$$

Recalling that $w = \partial_u(A\mathbf{r}_u + B\mathbf{r}_v) + \partial_v(B\mathbf{r}_u + C\mathbf{r}_v)$ and using the equations (3) for the second fundamental form components, we obtain

$$w = (A_u + B_v)\mathbf{r}_u + (B_u + C_v)\mathbf{r}_v + (eA + 2fB + gC)\mathbf{N} = H\,\mathbf{r}_u + I\,\mathbf{r}_v + K\,\mathbf{N}.$$

Hence we may compute

$$\begin{aligned} \langle w_u, \mathbf{r}_u\rangle &= \partial_u(\langle w, \mathbf{r}_u\rangle) - \langle w, \mathbf{r}_{uu}\rangle = H_u - eK, \\ \langle w_u, \mathbf{r}_v\langle &= \partial_u(\langle w, \mathbf{r}_v\rangle) - \langle w, \mathbf{r}_{uv}\rangle = I_u - fK, \\ \langle w_v, \mathbf{r}_v\rangle &= \partial_v(\langle w, \mathbf{r}_v\rangle) - \langle w, \mathbf{r}_{vv}\rangle = I_v - gK, \end{aligned}$$

and these tell us the left sides of (38). On the other hand, the right sides satisfy

$$\langle \mathbf{r}_{ttu}, \mathbf{r}_{tu}\rangle = \partial_t\langle \mathbf{r}_{tu}, \mathbf{r}_u\rangle - \langle \mathbf{r}_{tu}, \mathbf{r}_{tu}\rangle = -|\mathbf{r}_{tu}|^2,$$

which gives the right side of the first of equations (37) after using the tangent conditions for $\mathbf{r}_t$, c.f. (15) where $\mathbf{h} = \mathbf{r}_t$. Similarly the other right sides of (38) simplify to the remaining equations (37). $\square$

Assuming that we are able to (uniquely) solve for the functions H, I and K the geodesic equation is then given by:

$$\boxed{\mathbf{r}_{tt} = H\mathbf{r}_u + I\mathbf{r}_v + K\mathbf{N}}$$

In fact it will turn out that the special form of $\mathbf{r}_t$ will allows us to write the derive required solution for the functions H, I, and K satisfying (37). The remainder of the paper is concerned with the derivation of this solution formula, which is unfortunately rather technical.

4.4. Solving for H, K and I.

Proposition 2. *Suppose $t \mapsto \mathbf{r}(t, u, v)$ is a curve in the space of flags, where the components of $\mathbf{r}_t = a\mathbf{r}_u + b\mathbf{r}_v + c\mathbf{N}$ satisfy the conditions (21), and suppose that functions H, I, K satisfy the equations (37). Then in coordinates $x = \theta(u, v)$ and $y = v$, the solution is given by*

$$\begin{aligned} &H(x, y) = h(x) + yk(x), \qquad I(x, y) = m(x) + yn(x), \\ &K(x, y) = h'(x) + yk'(x) + [\beta'(x) - yJ'(x)]s(x)^2 + \frac{[\mu(x) + y\nu(x)]^2}{\beta'(x) - yJ'(x)}, \end{aligned} \tag{39}$$

where k and n are determined in terms of some functions h and m by

$$k(x) = 2J(x)s(x)^2 + \frac{J(x)h'(x) - m'(x)}{\beta'(x)} - \frac{2s'(x)\mu(x)}{\beta'(x)}, \tag{40}$$

$$n(x) = J(x)k(x) - \big[s(x)^2 + J(x)^2 s(x)^2 + s'(x)^2\big]. \tag{41}$$

The functions μ and ν are given in terms of the coefficients p, q, and s of (21) by

$$\begin{aligned} \mu(x) &= p''(x) + p(x) + J(x)q(x) \\ \nu(x) &= s''(x) + s(x) + J(x)^2 s(x). \end{aligned} \tag{42}$$

Proof. First we simplify the right sides of the equations (37). After changing to (x, y) coordinates and incorporating the tangent space constraints we have

$$\begin{aligned} \mathbf{r}_{tu} &= e(b_x - Jc)\mathbf{r}_v + e(c_x + a + Jb)\mathbf{N} \\ \mathbf{r}_{tv} &= -e(b_x - Jc)\mathbf{r}_u + \big(c_y + f(c_x + a + Jb)\big)\mathbf{N}. \end{aligned}$$

Using the explicit formulas from (21) together with the formula (20) for $e(x, y)$, these equations simplify to

$$\begin{aligned} \mathbf{r}_{tu} &= -s(x)\mathbf{r}_v + \frac{\mu(x) + y\nu(x)}{\beta'(x) - yJ'(x)}\mathbf{N} \\ \mathbf{r}_{tv} &= s(x)\mathbf{r}_u + \Big(s'(x) + J(x)\,\frac{\mu(x) + y\nu(x)}{\beta'(x) - yJ'(x)}\Big)\mathbf{N}. \end{aligned} \tag{43}$$

In (x, y) coordinates, the equations (37) become

$$H_x - K = -|\mathbf{r}_{tu}|^2/e \tag{44}$$

$$fH_x + H_y + eI_x - 2fK = -2\mathbf{r}_{tu} \cdot \mathbf{r}_{tv}, \tag{45}$$

$$fI_x + I_y - gK = -|\mathbf{r}_{tv}|^2. \tag{46}$$

We view (44) as determining K in terms of H, and plugging this into (45)–(46), we obtain the system

$$H_y + eI_x - fH_x = 2J|\mathbf{r}_{tu}|^2 - 2\mathbf{r}_{tu}\cdot\mathbf{r}_{tv} \tag{47}$$

$$I_y + fI_x - gH_x = J^2|\mathbf{r}_{tu}|^2 - |\mathbf{r}_{tv}|^2. \tag{48}$$

Multiplying (47) by J and subtracting from (48), we obtain

$$\frac{\partial}{\partial y}\big(I(x,y) - J(x)H(x,y)\big) = -|J\mathbf{r}_{tu} - \mathbf{r}_{tv}|^2 = -\big(s(x)^2 + J(x)^2 s(x)^2 + s'(x)^2\big),$$

and we conclude that

$$I(x,y) - J(x)H(x,y) + \big(s(x)^2 + J(x)^2 s(x)^2 + s'(x)^2\big)y + \xi(x) \tag{49}$$

for some function $\xi(x)$.

Now use (49) in (47) to eliminate I, and we obtain using (43)

$$\begin{aligned} H_y + \frac{J'(x)}{\beta'(x) - yJ'(x)}H(x,y) &= e(x,y)y\,\frac{d}{dx}\Big(s(x)^2 + J(x)^2 s(x)^2 + s'(x)^2\Big) \\ &\quad + e(x,y)\xi'(x) + 2J(x)s(x)^2 - 2s'(x)e(x,y)\mu(x) \\ &\qquad - 2s'(x)e(x,y)y\nu(x). \end{aligned}$$

The important thing here is that $\nu(x)$ mostly cancels with the $\frac{d}{dx}$ term – see (42) – to eliminate the y dependence, and we end up with

$$H_y + \frac{J'(x)}{\beta'(x) - yJ'(x)}\,H = \frac{-2s'(x)\mu(x) + \xi'(x) + 2J(x)s(x)^2\beta'(x)}{\beta'(x) - yJ'(x)}.$$

We conclude that $H(x,y) = h(x) + yk(x)$ for some functions h and k, and (49) implies that $I(x,y) = m(x) + yn(x)$ for some functions m and n, as in (39).

We now start over with the functions h, k, m, and n of the single variable x on $[0,L]$. Equation (49) immediately implies (41) upon matching the terms involving a y. Meanwhile (47) give (40) upon matching the terms that do not depend on y. All other equations are simply derivatives of these, so the equations h and m are arbitrary.

□

At long last, it remains to figure out the two arbitrary functions h and m using the boundary conditions from Lemma 6. To do this, we need to solve equations (36) for A, B, and C, and ensure that the boundary conditions (24)–(27) are satisfied.

Lemma 8. *Let A, B, and C satisfy the equations (36), where H, I, and K satisfy the conditions in Proposition 2. Define $P = A + JB$ and $Q = B + JC$. Then we have*

$$Q(x,y) = \frac{Q_0(x) + \int_0^y \Psi(x,s)\,ds}{[\beta'(x) - yJ'(x)]^2} \tag{50}$$

for some function $Q_0(x)$, where the integrand Ψ is given by

$$\begin{aligned}\Psi(x,y) &= [\beta'(x) - yJ'(x)]^2\big[H(x,y) + J(x)I(x,y)\big] \\ &\qquad - [\beta'(x) - yJ'(x)]\,\frac{\partial}{\partial x}\big([\beta'(x) - yJ'(x)]K(x,y)\big).\end{aligned}$$

Proof. In terms of P and Q, equations (36) yield, in (x,y) variables,

$$P + JQ = K/e \tag{51}$$

$$eP_x - eJ'(x)B + B_y = H \tag{52}$$

$$eQ_x - eJ'(x)C + C_y = I \tag{53}$$

Combining (52)–(53), we get

$$e\frac{\partial}{\partial x}(P+JQ) - eJ'Q - eJ'(B+JC) + (B+JC)_y = H + JI,$$

which reduces to

$$\begin{aligned} Q_y(x,y) &- \frac{2J'(x)}{\beta'(x) - yJ'(x)}Q(x,y) \\ &= H(x,y) + J(x)I(x,y) - e\frac{\partial}{\partial x}\big(K(x,y)/e(x,y)\big) \\ &= H(x,y) + J(x)I(x,y) - \frac{\frac{\partial}{\partial x}\big([\beta'(x) - yJ'(x)]K(x,y)\big)}{\beta'(x) - yJ'(x)}. \end{aligned}$$

Multiplying both sides by the integrating factor $[\beta'(x) - yJ'(x)]^2$ yields the ODE

$$\frac{\partial}{\partial y}\Big(\big[\beta'(x) - yJ'(x)\big]^2 Q(x,y)\Big) = \Psi(x,y). \tag{54}$$

which can be solved by (50). □

As an obvious consequence of (51), we obtain

$$P(x,y) = [\beta'(x) - yJ'(x)]K(x,y) - J(x)Q(x,y),$$

where Q is given by (50). To finish off, we need to find A, B, and C. It is sufficient to determine C, since $B = Q - JC$ and $A = P - JB$. We have:

Lemma 9. *Suppose A, B, and C are as in Lemma 8. Then*

$$C(x,y) = \frac{C_0(x) + \int_0^y [\beta'(x) - sJ'(x)]I(x,s)\,ds - \frac{\partial}{\partial x}\int_0^y Q(x,s)\,ds}{\beta'(x) - yJ'(x)} \tag{55}$$

for some function $C_0(x)$.

Proof. This is just a matter of solving (53) for C. It may be rewritten as

$$-J'(x)C(x,y) + [\beta'(x) - yJ'(x)]C_y(x,y) = [\beta'(x) - yJ'(x)]I(x,y) - Q_x(x,y), \tag{56}$$

and the term on the left is already a y-derivative. So we integrate both sides with respect to y, then observe that the x derivative on Q may be pulled outside the y integral to get (55). □

At this point we have completely solved for A, B, and C, with two arbitrary functions $Q_0(x)$ and $C_0(x)$ appearing in the solution. However it is not hard to see that those functions are "invisible" to the flag, in the sense that the decomposition of Lemma 7 only sees the vector field $w = H\mathbf{r}_u + I\mathbf{r}_v + K\mathbf{N}$, not the components A, B, and C that generate them through (36). Another way of thinking of this is that the PDE system (36), with homogeneous right sides, has nontrivial solutions parameterized by two arbitrary functions, which we may view as $C_0(x)$ and $Q_0(x)$. Thus we expect that the boundary conditions (24)–(27) should not involve C_0 and Q_0 at all. Only the two functions $h(x)$ and $m(x)$ given in Proposition 2 are constrained, since those are "visible" to the flag. This is observation is demonstrated by our final result on the resulting boundary conditions:

Proposition 3. *If H, I, and K are defined in terms of A, B, and C by (36), and satisfy the conditions of Proposition 2, then the boundary conditions of Lemma 6 take the form*

$$\int_0^1 \Psi(x,y)\,dy = 0 \tag{57}$$

$$-\frac{d}{dx}\left(\int_0^1 \frac{y\,\Psi(x,y)\,dy}{\beta'(x)[\beta'(x)-yJ'(x)]}\right) = \int_0^1 [\beta'(x)-yJ'(x)]I(x,y)\,dy \tag{58}$$

for $x \in [0,\theta^*]$, and

(59)

$$J(x)\int_0^{\eta(x)} \Psi(x,y)\,dy = [\beta'(x)-yJ'(x)]^3 K(x,\eta(x))$$

$$\frac{d}{dx}\left(-\int_0^{\eta(x)} \frac{y\,\Psi(x,y)\,dy}{\beta'(x)[\beta'(x)-yJ'(x)]} + \frac{\eta(x)[\beta'(x)-yJ'(x)]}{J(x)\beta'(x)}K(x,\eta(x))\right)$$

$$= \int_0^{\eta(x)} [\beta'(x)-yJ'(x)]I(x,y)\,dy$$

for $x \in [\theta^*, L]$.

Proof. In terms of the functions $P = A + JB$ and $Q = B + JC$, equation (26) becomes

$$[\beta'(x)-yJ'(x)]^2 Q(x,1) - \beta'(x)^2 Q(x,0) = 0,$$

and by (54) we conclude (57). Since (56) may be written as

$$\tilde{C}_y(x,y) = [\beta'(x)-yJ'(x)]I(x,y) - Q_x(x,y),$$

equation (34) becomes

$$\int_0^1 [\beta'(x)-yJ'(x)]I(x,y)\,dy - \int_0^1 Q_x(x,y)\,dy + \int_0^1 \psi'(x)\,dy = 0,$$

which simplfies to

$$\frac{d}{dx}\left(\int_0^1 Q(x,y)\,dy - \frac{\beta'(x)-yJ'(x)}{\beta'(x)}Q(x,1)\right) = \int_0^1 [\beta'(x)-yJ'(x)]I(x,y)\,dy$$

using (33) for $\psi(x)$. Using formula (50) for $Q(x,y)$, we can show that the left side of this equation is the left side of (58), after applying (57) to simplify the integral of $\Psi(x,y)$. This establishes both conditions on $[0,\theta^*]$.

The conditions on $[\theta^*, L]$ are a bit more involved. Equation (26) can be written, using (51) to eliminate P, as

$$J(x)\Big([\beta'(x)-\eta(x)J'(x)]^2 Q(x,\eta(x)) - \beta'(x)^2 Q(x,0)\Big) \\ = [\beta'(x)-\eta(x)J'(x)]^3 K(x,\eta(x)),$$

and the left side of this equation is the left side of (59) using equation (54), as before. Finally equation (35) takes the form

$$\tilde{B}(x,\eta(x))/J(x) + \tilde{C}(x,0) + \frac{d}{dx}\left(\frac{\eta(x)[\beta'(x)-yJ'(x)]P(x,\eta(x))}{J(x)\beta'(x)}\right) = 0.$$

Now we use $B = Q - JC$ and $P = K/e - JQ$ to simplify this, and obtain

$$\frac{[\beta'(x)-\eta(x)J'(x)]Q(x,\eta(x))}{J(x)} - \tilde{C}(x,\eta(x)) + \tilde{C}(x,0) \\ + \frac{d}{dx}\left(\frac{\eta(x)[\beta'(x)-yJ'(x)]^2 K(x,\eta(x))}{J(x)\beta'(x)} - \frac{\eta(x)[\beta'(x)-yJ'(x)]Q(x,\eta(x))}{\beta'(x)}\right) = 0.$$

□

These equations now represent ODEs for the functions h and m, and solving them is analogous to solving the ODE for the tension σ in (1). We do not see any further simplifications to be obtained, but the formula is relatively easy to obtain as functions like $\Psi(x,y)$ depend on y only as a third-order polynomial (so that the integration is basically trivial). In the special case that $J \equiv 0$ so that $f = g \equiv 0$, these equations reduce to the inextensible threads equation (1).

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