NONPOSITIVE CURVATURE OF THE QUANTOMORPHISM GROUP AND QUASIGEOSTROPHIC MOTION

JAE MIN LEE

Department of Mathematics, KTH Royal Institute of Technology, 100 44 Stockholm, Sweden

STEPHEN C. PRESTON

Department of Mathematics, Brooklyn College and the Graduate Center, City University of New York, NY 11106, USA

Abstract. In this paper, we compute the sectional curvature of the group whose Euler-Arnold equation is the quasi-geostrophic (QG) equation in geophysics and oceanography, or the Hasegawa-Mima equation in plasma physics: this group is a central extension of the quantomorphism group $\mathcal{D}_q(M)$. We consider the case where the underlying manifold M is rotationally symmetric, and the fluid flows with a radial stream function. Using an explicit formula for the curvature, we will also derive a criterion for the curvature operator to be nonpositive and discuss the role of the Froude number and the Rossby number on curvature. The main technique to obtain a usable formula is a simplification of Arnold's general curvature formula in the case where a vector field is close to a Killing field, and then use the Green's function explicitly together with a criterion for nonnegativity of a general bilinear form. We show that nonzero Froude number and Rossby numbers typically both tend to stabilize flows in the Lagrangian sense, although there are counterexamples in general.

The Euler equation on a surface M for a divergence-free velocity field $X = \mathrm{sgrad} f$ is given by

$$\Delta f_t + \{f, \Delta f\} = 0, \qquad f(0,x) = f_0(x).$$

Together with the flow equation $\frac{\partial \eta}{\partial t} = X \circ \eta$, this defines a geodesic in the group of volume-preserving diffeomorphisms $\mathrm{Diff}_\mu(M)$ under the right-invariant L^2 kinetic energy Riemannian metric, and thus we can compute sectional curvatures of this group to determine stability of Lagrangian perturbations. Arnold [1] computed this in the case $M = \mathbb{T}^2$ with $f(x,y) = \cos kx$ for $k \in \mathbb{N}$, showing that $\langle R(Y,X)X, Y\rangle \le 0$ for every $Y \in T_{\mathrm{id}}\mathrm{Diff}_\mu(\mathbb{T}^2)$; in other words the curvatures of all planar sections containing the vector X are nonpositive. This indicates that all Lagrangian perturbations of the steady velocity field $X = -k \sin kx\, \partial_y$ grow at least linearly in time.

This work was generalized by Misiołek [9] who showed that $\langle R(Y,X)X, Y\rangle \le 0$ for all Y whenever X is of the form $X = u(x)\, \partial_y$ for any u. Surveys of similar work on the sign of the sectional curvature for other diffeomorphism groups with other right-invariant metrics are given in the book by Arnold-Khesin [2] and the more recent survey paper [7]. The recent paper [15] studies this problem on the

E-mail addresses: `lee9@kth.se, stephen.preston@brooklyn.cuny.edu`.

Date: December 3, 2020.

2010 *Mathematics Subject Classification.* 35Q35, 53D25.

Key words and phrases. Quantomorphism group, Quasi-Geostrophic equation, Hasegawa-Mima equation, Nonpositive sectional curvature.

group of axisymmetric volume-preserving diffeomorphisms of a solid torus, classifying those X for which $\langle R(X,Y)Y,X\rangle > 0$ for all Y. In particular we note the second author's paper [12], which classified the steady velocity fields X (i.e., those arising from f satisfying $\{f,\Delta f\}=0$) such that $\langle R(Y,X)X,Y\rangle \le 0$ for all Y. The present paper generalizes this result.

We consider a two-dimensional central-extension of the group $\mathrm{Diff}_\mu(M)$: the first dimension corresponds to using the full stream function $f\colon M\to\mathbb{R}$ and not merely its skew-gradient, while the second dimension is an additional real parameter as arises in Vizman [13]. Vectors in the Lie algebra $\mathfrak{G}$ take the form $X=(f,\beta)$ for $\beta\in\mathbb{R}$, and if $Y=(g,\gamma)$ is another such vector, then the Lie bracket is defined by

$$[X,Y] = \Big(\{f,g\}, \int_M \chi\{f,g\}\,dA\Big) \tag{1}$$

for some function χ, while the Riemannian metric at the identity is given by

$$\langle X,Y\rangle = \int_M \alpha^2 fg + \langle\nabla f,\nabla g\rangle\,dA + \beta\gamma. \tag{2}$$

The Lie algebra determines the Lie group, and the metric is extended by right-translation to this group. The geodesic equation then splits into the flow equation and the Euler-Arnold equation, here given by

$$\alpha^2 f_t - \Delta f_t - \{f,\Delta f\} - \beta f_\theta = 0, \qquad \beta_t = 0. \tag{3}$$

This is called the Hasegawa-Mima equation or the quasigeostrophic equation (not to be confused with the unrelated surface quasigeostrophic equation), and the parameters α and β represent correction effects as detailed below.

Theorem 1. *Suppose M has rotationally symmetric metric $ds^2 = dr^2 + \varphi(r)^2\,d\theta^2$ for $\theta\in S^1$ and $r\in[0,R]$, where $\varphi>0$ on $(0,R)$. Let $X = \big(f(r),\beta\big)$ for some function $f\colon[0,R]\to\mathbb{R}$ and some $\beta\in\mathbb{R}$; assume that f and its derivatives have only isolated zeroes. Consider X as a steady solution of the Euler-Arnold equation* (3) *on the group G with Lie bracket* (1) *and right-invariant metric* (2)*, and let $u(r)=\frac{f'(r)}{\varphi(r)}$ denote the corresponding velocity field profile. Then the sectional curvature in all sections containing X is nonnegative iff the function Q given by*

$$Q := \frac{\frac{d}{dr}(\varphi' u) - \frac12\varphi(\beta+\alpha^2 u)}{u'} \tag{4}$$

is well-defined everywhere (i.e., u' is nowhere zero without the numerator also being zero) and satisfies the differential inequality

$$\varphi(r)Q'(r) + Q(r)^2 \le \alpha^2\varphi(r)^2 + 1 \tag{5}$$

for every $r\in[0,R]$.

The main differences between our approach here and the approach of [12] are as follows: first, we cannot use the submanifold geometry as in [9], so we need to use the general Arnold formula for sectional curvature, which we rewrite in the more convenient form (19). We also avoid the assumption that u is real-analytic and present simpler and cleaner proofs, particularly of the criterion in Theorem 12 for a bilinear form of the type

$$B(g,g) = 2\int_0^R\int_0^r \xi_1(r)\xi_0(s)\mathrm{Re}\big(\overline{g(s)}g(r)\big)\,ds\,dr$$

to be nonnegative. Because our differential operator $(\alpha-\Delta)$ is more complicated, we do not have an explicit Green function, but we show that this is not necessary to perform the computations.

We discuss the background of this equation in Section 1; the rest of the paper is devoted to computing the curvature in order to prove Theorem 1, and we conclude with a couple of examples in Section 4 to illustrate the effect of the parameters α and β.

1. Background

There are two classical viewpoints on the motion of a fluid. First, the Eulerian perspective concerns $u(t,x)$, the velocity of a fluid particle located at the point x at time t, and one studies the evolution equation of u with the prescribed initial/boundary conditions. In the Lagrangian formalism, one considers the function $\eta(t,x)$, which is the position at time t of a fluid particle which at time zero was at x. So one can think of the collection of $\eta(t,\cdot)$ as giving the configuration of the particles at each time t and can recover the Eulerian description via $u(t,\cdot) = \eta_t \circ \eta^{-1}$. In the case of ideal fluid on a Riemannian manifold M, the configuration space is $\mathcal{D}_\mu(M)$, the group of volume preserving diffeomorphisms on M where μ is the volume form on M. In his beautiful paper in 1966, Arnold [1] observed that the Euler equation for ideal fluid can be realized as the geodesic equation on $\mathcal{D}_\mu(M)$ endowed with the right-invariant kinetic energy metric, and this observation was rigorously justified by Ebin and Marsden in 1970 [4]. Since then, geodesic equations on the diffeomorphism groups endowed with an invariant metric have been studied extensively. Invariance leads to a reduction of order to a first-order equation on the Lie algebra, which is called the Euler-Arnold equation.

The quasi-geostrophic equation (QG) describes large scale flows in atmosphere and ocean which have large horizontal to vertical aspect ratio. Here, quasi-geostrophy means that Coriolis force and horizontal pressure gradient forces are nearly in balance, which allows the momentum equation for the flow to be prognostic and include nonlinear dynamics. In terms of the stream function $\psi(t,x,y)$ of the velocity u of the barotropic fluid, the QG equation in the β-plane approximation is given by

$$\partial_t\left(\Delta\psi - \alpha^2\psi\right) + \{\psi, \Delta\psi\} + \beta\psi_x = 0, \tag{6}$$

where α^2 denotes the Froude number and β is the Rossby number, the gradient for the Coriolis parameter. Here, $\{\cdot,\cdot\}$ is the Poisson bracket, i.e., $\{g,h\} = h_y g_x - g_x h_y$. The Coriolis parameter f is approximated in the β-plane by $f = f_0 + \beta y$ with constants f_0 and β. The case when $\beta = 0$ is the f-plane approximation. The Froude number α^2 is a nondimensionalized parameter defined by

$$\alpha := \frac{u_0}{\sqrt{g_0 l_0}},$$

where u_0 is the velocity scale, g_0 is the gravitational constant, and l_0 is the horizontal length scale. So α measures the effect of gravity and $\alpha \ll 1$ in the mesoscale motions of the atmosphere and oceans in the midlatitudes. Additionally for α and β both nonzero, equation (6) is the Hasegawa-Mima equation arising in plasma dynamics [16]. The equation (6) can also be written in terms of the potential vorticity as

$$\partial_t\omega + \{\psi,\omega\} = 0, \qquad \omega = \Delta\psi - \alpha^2\psi + \beta y, \tag{7}$$

which is similar to the vorticity-stream formulation of the 2-dimensional incompressible Euler equation. The QG equation can be derived as the inviscid limit of the rotating shallow-water equations, as well. For more mathematical theory of atmospheric and oceanic fluid, see Majda [8]. For more comprehensive background on the geostrophical fluid dynamics, see Pedlosky [10]. It is important to note that equation (7) is *not* the "surface quasi-geostrophic" (SQG) equation; the SQG equation is when $\omega = \sqrt{-\Delta}\psi$, and it has completely different properties. See [3] and [14] for the geometric approach to SQG, and references therein for other aspects.

From the geometric point of view, the QG equation is of interest since it is an example of the Euler-Arnold equation. In 1994, Zeitlin-Pasmanter [16] showed that the QG equation can arise as the Euler-Arnold equation in the infinite dimensional Lie algebra and its central extension, without constructing the full group. They also computed the sectional curvature and showed that it is negative in the section spanned by the cosinusoidal stationary flows. In 1998, Holm-Zeitlin [6] showed that the QG equation in the f- and β-plane approximations are the geodesic equations on the group of symplectic diffeomorphisms by using variational principles for QG dynamics. Also, in 2008, Vizman [13] showed that the equation (6) is the Euler-Arnold equation on the central extension of the group of Hamiltonian diffeomorphisms in the case when $\alpha = 0$. Finally, Ebin-Preston [5] showed in 2015 that the QG equation is the geodesic equation on a central extension of the quantomorphism group (thus constructing the group corresponding to the Lie algebra in [16]).

On a contact manifold (M, θ), the quantomorphism group $\mathcal{D}_q(M)$ is defined as the space of diffeomorphisms on M that preserve the contact form θ exactly. So the quantomorphisms group is a subgroup of the contactomorphism group $\mathcal{D}_\theta(M)$, whose elements preserve the contact structure, i.e., $\eta^*\theta = e^\lambda\theta$ for some $\lambda\colon M \to \mathbb{R}$. If the contact form is regular, then $\mathcal{D}_\theta(M)$ is related to a symplectic manifold by a Boothby-Wang fibration and the tangent space of $\mathcal{D}_q(M)$ can be identified with the space of functions $f\colon M \to \mathbb{R}$ such that $E(f) = 0$, where E is the Reeb field. Furthermore, one can show that $\mathcal{D}_q(M) \subset \mathcal{D}_\theta(M)$ is a totally geodesic submanifold. For more Riemannian geometry of the contactomorphism group in general, see Ebin-Preston [5].

Let M be a 2-dimensional manifold with symplectic form ω (a nowhere-zero 2-form). On top of M, there is a 3-dimensional manifold N with a contact form ϕ such that $\phi \wedge d\phi$ is nowhere-zero, and a projection map $\pi\colon N \to M$ satisfying $\pi^*\omega = d\phi$. Recall that for the contact form ϕ, there is a unique vector field E, called the Reeb field, satisfying the two conditions $\phi(E) = 1$ and $\iota_E d\phi = 0$. In the simplest case M is the flat cylinder $M = [0, R] \times \mathbb{S}^1$ with $N = [0, R] \times \mathbb{S}^1 \times \mathbb{S}^1$, where $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, with $\phi = dz - y\,dx$ and $\omega = dx \wedge dy$. In this case, the Reeb field is $E = \partial_z$.

The space of quantomorphisms $\mathcal{D}_q(N)$ consists of diffeomorphisms η on N that preserve the contact form exactly, i.e., $\eta^*\phi = \phi$. Its tangent space at the identity consists of vector fields X such that $\mathcal{L}_X\phi = 0$, and such a vector field X is uniquely determined by the function $\psi = \phi(X)$ via the formula $\iota_X d\phi + d\psi = 0$, and we can write $X = S_\phi\psi$, following [5]. That is, S_ϕ is a Lie algebra homomorphism. In the case with $\phi = dz - y\,dx$, we have

$$X = S_\phi\psi = -\psi_y\,\partial_x + \psi_x\,\partial_y + (\psi + y\psi_y)\,\partial_z.$$

This preserves the contact form iff $\psi_z = 0$, and conversely any such function with $\psi_z = 0$ gives a quantomorphism vector field. That is, we can identify elements $X \in T_{\mathrm{Id}}\mathcal{D}_q(M)$ with E-invariant functions on N, which are identified with all functions on M. In this way the one-dimensional trivial central extension of $\mathrm{Diff}_\mu(M)$ is interpreted as the group $\mathrm{Diff}_q(N)$. We will not need this machinery for the present situation however; it is sufficient to know the formulas (1) and (11).

As in the finite dimensional Lie group case, the sectional curvature of the diffeomorphism group provides information about the stability of geodesics, which we call the Lagrangian stability. For example, positive curvature in all sections implies that geodesics with close initial data locally converge (stability) while negative sectional curvature implies that the geodesics spread apart (instability). Eulerian and Lagrangian stability are different but related: for example if a fluid is stable in the Eulerian sense, then the linearized Lagrangian perturbations can grow at most polynomially in time; see the second author's paper [11]. For more discussions on the curvature of the Euler-Arnold equations in general, see Khesin et al. [7].

2. The coadjoint computation

On a domain M described in (r, θ) coordinates by $0 \le r \le R$ and $\theta \in S^1$, we assume the metric is given by

$$ds^2 = dr^2 + \varphi(r)^2\,d\theta^2$$

for a function φ which is positive on $(0, R)$ and may collapse to zero at $r = 0$ or at both $r = 0$ and $r = R$ (due to a coordinate singularity).

- If $\varphi(0) > 0$ and $\varphi(R) > 0$, e.g., $\varphi(r) = 1$, then M is an annulus.
- If $\varphi(0) = 0$ and $\varphi(R) > 0$, e.g., $\varphi(r) = r$, then M is a disc.
- If $\varphi(0) = 0$ and $\varphi(R) = 0$, e.g., $\varphi(r) = \sin r$ for $R = \pi$, then M is a sphere.

We assume that if $\varphi(0) = 0$ then $\varphi'(0) = 1$, while if $\varphi(R) = 0$ then $\varphi'(R) = -1$. This ensures that the metric is locally Euclidean.

The area element is then $dA = \varphi(r)\,dr\,d\theta$, and the Poisson bracket of functions $f, g\colon M \to \mathbb{R}$ is

$$\{f, g\} = \frac{1}{\varphi(r)} \left(\frac{\partial f}{\partial r}\frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial r} \right). \tag{8}$$

We construct a function

$$\chi\colon (0, R) \to \mathbb{R}, \text{ such that } \chi'(r) = \varphi(r). \tag{9}$$

The Lie algebra will then be defined in terms of stream functions $f\colon M \to \mathbb{R}$. Such functions must be constant on the boundary components $r = 0$ and $r = R$ (if φ is nonzero there), although we may have two different constants: this ensures that the Hamiltonian velocity field is tangent to the boundary.

Definition 2. For functions $f, g\colon M \to \mathbb{R}$ which are constant on the boundary (if any) of M, and real numbers β, γ, let $X = (f, \beta)$ and $Y = (g, \gamma)$ be vectors. We define a Lie bracket by the formula

$$[X, Y] = \Big(\{f, g\}, \int_M \chi\{f, g\}\,dA \Big). \tag{10}$$

We denote the Lie algebra by $\mathfrak{G}$.

That equation (10) does indeed define a Lie algebra is easy to check: antisymmetry is obvious, and the Jacobi identity for the Lie bracket is an easy consequence of the Jacobi identity for the Poisson bracket. Finally the fact that $\{f, g\}$ is constant on the boundary components $r = \text{constant}$ follows directly from the formula (8). The Lie algebra $\mathfrak{G}$ thus defined is a two-dimensional central extension of the Lie algebra of Hamiltonian vector fields, since $[\text{sgrad} f, \text{sgrad} g] = \text{sgrad}\{f, g\}$.

We define an inner product on the Lie algebra $\mathfrak{G}$ by

$$\langle (f, \beta), (g, \gamma) \rangle = \int_M \Big(\alpha^2 fg + \langle \nabla f, \nabla g \rangle \Big)\,dA + \beta\gamma. \tag{11}$$

From the Lie bracket (10) and inner product (11), we determine the ad-star operator, upon which all the geometry depends.

Proposition 3. *Let f, g be functions on M which are constant on the boundary (if any) of M, and let $\beta, \gamma \in \mathbb{R}$. Set $X = (f, \beta)$ and $Y = (g, \gamma)$. If $\alpha \neq 0$, then the ad-star operator defined by the condition*

$$\langle \text{ad}^\star{}_X Y, Z \rangle = -\langle [X, Z], Y \rangle \qquad \textit{for every } Z \in \mathfrak{G}$$

is given explicitly by $\text{ad}^\star{}_X Y = (j, 0)$, where j is a function given in terms of the Poisson bracket (8) and the function χ given by (9), by the following conditions:

$$\alpha^2 j - \Delta j = \{f, \alpha^2 g + \gamma\chi - \Delta g\}, \tag{12}$$

and on any boundary component, j is constant and j_r integrates to zero.

This j exists and is unique unless $\alpha = 0$, in which case it is unique only up to a constant.

Proof. We first note that the integral of any Poisson bracket $\{f, g\}$ on M is zero, using the fact that

$$\begin{aligned}\int_M \{f, g\}\,dA &= \int_0^R \int_0^{2\pi} \big(f_r g_\theta - f_\theta g_r \big)\,d\theta\,dr = -\int_0^{2\pi}\int_0^R \frac{\partial}{\partial r}\Big(g\,\frac{\partial f}{\partial \theta} \Big)\,d\theta\,dr \\ &= -\int_0^{2\pi} g\,\frac{\partial f}{\partial \theta}\,d\theta \Big|_{r=0}^{r=R} = 0,\end{aligned}$$

since f is constant on both boundary components.

Now let $Z = (h, \delta)$ be another vector in $\mathfrak{G}$ with h constant on the boundary. Then we compute using the definition:

$$\begin{aligned}\langle \mathrm{ad}^\star{}_X Y, Z\rangle &= -\langle [X, Z], Y\rangle = -\Big\langle \Big(\{f, h\}, \int_M \chi\{f, h\}\, dA\Big), (g, \gamma)\Big\rangle \\ &= -\int_M \alpha^2\{f, h\} g\, dA - \int_M \langle \nabla\{f, h\}, \nabla g\rangle\, dA - \gamma \int_M \chi\{f, h\}\, dA \\ &= -\int_M \{f, h\}(\alpha^2 g + \gamma\chi - \Delta g)\, dA - \int_{\partial M} \{f, h\}\partial_r g\varphi\, d\theta,\end{aligned}$$

by the Divergence Theorem. However since f and h are both constant on any boundary components $r = 0$ or $r = R$, we see from formula (8) that $\{f, h\}$ must in fact be zero on the boundary. So this boundary integral vanishes in any case.

Using the Leibniz rule $\{f, h\}q = \{f, hq\} - \{f, q\}h$, and the fact that Poisson brackets integrate to zero, we get

$$\langle \mathrm{ad}^\star{}_X Y, Z\rangle = \int_M \{f, q\}h\, dA, \qquad \text{where } q = \alpha^2 g + \gamma\chi - \Delta g. \tag{13}$$

On the other hand, if $\mathrm{ad}^\star{}_X Y = (j, \epsilon)$ for some function $j\colon M \to \mathbb{R}$ and $\epsilon \in \mathbb{R}$, then we would have

$$\langle \mathrm{ad}^\star{}_X Y, Z\rangle = \int_M (\alpha^2 j - \Delta j)h\, dA + \int_{\partial M} \varphi h \partial_r j\, d\theta + \delta\epsilon. \tag{14}$$

The boundary term simplifies, since both h and φ are constant on the boundary, to

$$\int_{\partial M} \varphi h\partial_r j\, d\theta = h(R)\varphi(R) \int_{S^1} j_r(R, \theta)\, d\theta - h(0)\varphi(0) \int_{S^1} j_r(0, \theta)\, d\theta.$$

Since (14) must equal the right side of (13) for every choice of constant δ and function h constant on the boundary, we see that $\epsilon = 0$, while j satisfies

$$\alpha^2 j - \Delta j = \{f, \alpha^2 g + \gamma\chi - \Delta g\}, \tag{15}$$

with boundary conditions that j is constant on the boundary components and $\int_{S^1} j_r\, d\theta = 0$ for $r = 0$ or $r = R$. These boundary conditions only apply when $\varphi \neq 0$ at either $r = 0$ or $r = R$.

To finish off the proof, we need to verify that there actually is such a function j satisfying these boundary conditions. In general, when boundaries are involved, it is not guaranteed that there even is an operator $\mathrm{ad}^\star{}_X Y$ for any given Lie algebra and inner product. If the right side of (15) is written in a Fourier series as $p(r, \theta) = \sum_{n\in\mathbb{Z}} p_n(r)e^{in\theta}$, and we similarly expand j, then the functions $j_n(r)$ satisfy

$$-\frac{1}{\varphi(r)}\frac{d}{dr}\big(\varphi(r)j_n'(r)\big) + \left(\alpha^2 + \frac{n^2}{\varphi(r)^2}\right) j_n(r) = p_n(r).$$

The boundary conditions become $j_n(0) = 0$ and $j_n(R) = 0$ for $n \neq 0$, and the usual results for ODEs with boundary conditions show that there is a unique solution $j_n(r)$. If we have $\varphi(0) = 0$ or $\varphi(R) = 0$, then requiring that $j_n(0)$ or $j_n(R)$ respectively remain finite also ensures a unique solution. Finally for $n = 0$ the Neumann boundary condition kicks in to give $j_0'(0) = j_0'(R) = 0$, which has a unique solution as long as $\alpha \neq 0$. When $\alpha = 0$ and $n = 0$, there is a nonunique solution for j_0 (and thus j is only unique up to a constant). The solution does exist as a consequence of the fact that $\int_M p\, dA = 0$ since p is a Poisson bracket. □

Remark 4. If $\alpha = 0$, then the inner product (11) is degenerate on the space of all functions (e.g., when $\chi \equiv 0$). In this case it is only defined on the space of gradients of functions, and constants are killed, which is the situation when we deal with divergence-free vector fields. This makes things slightly more complicated when we try to obtain limiting cases when $\alpha = 0$ from the other results, since we genuinely lose a dimension (corresponding to an overall constant in a stream function, e.g., the integral of it over the manifold). For the most part this does not matter, but we must be cautious about it.

The Euler-Arnold equation on the corresponding Lie group, where the inner product (11) is extended by right-translations to a right-invariant Riemannian metric, can now be computed using the general formula $\frac{dX}{dt} + \mathrm{ad}^\star{}_X X = 0$; see [2] or [7].

Corollary 5. *The Euler-Arnold equation on the Lie algebra* $\mathfrak{G}$ *with inner product* (11) *is given by*

$$\alpha^2 f_t - \Delta f_t - \{f, \Delta f\} - \beta f_\theta = 0, \qquad \beta_t = 0. \tag{16}$$

Proof. The formula (12) says (with $g = f$ and $\gamma = \beta$) that

$$f_t + j = 0, \qquad \text{where } \alpha^2 j - \Delta j = \{f, \alpha^2 f + \beta\chi - \Delta f\}, \tag{17}$$
$$\text{such that } j \text{ is constant and } j_r \text{ integrates to zero on each boundary component.}$$

Here we note that $\{f, \alpha^2 f\}$ vanishes, while $\{f, \chi\} = -\frac{1}{\varphi(r)}\frac{\partial f}{\partial \theta}\chi'(r) = -\frac{\partial f}{\partial \theta}$. Hence equation (17) becomes (16) upon applying the operator $(\alpha^2 - \Delta)$ to both sides of $f_t + j = 0$. □

Note that (16) is simpler and reduces to the 2D Euler equation in case $\alpha = \beta = 0$, but it is not fully deterministic: equation (17) is needed to determine the evolution of f on the boundary of M. For us this will not matter, since we only need the following obvious consequence.

Corollary 6. *If* $f = f(r)$ *is any radial function, then* f *is a steady solution of the Euler-Arnold equation* (16)*.*

3. Application of the curvature formula

Recall that Arnold's curvature formula is (see [2] or [7])

$$\langle R(X,Y)Y, X\rangle = \tfrac{1}{4}\Big(|\mathrm{ad}^\star{}_X Y + \mathrm{ad}^\star{}_Y X|^2 + 2\langle \mathrm{ad}_X Y, \mathrm{ad}^\star{}_Y X - \mathrm{ad}^\star{}_X Y\rangle \\ - 3|\mathrm{ad}_X Y|^2 - 4\langle \mathrm{ad}^\star{}_X X, \mathrm{ad}^\star{}_Y Y\rangle\Big). \tag{18}$$

The sectional curvature of the plane σ spanned by X and Y is given by

$$K(\sigma) = \frac{\langle R(X,Y)Y, X\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2}.$$

A slightly simpler version of this formula can easily be obtained and will prove more convenient for our calculations.

$$\langle R(X,Y)Y, X\rangle = \tfrac{1}{4}|\mathrm{ad}^\star{}_X Y + \mathrm{ad}^\star{}_Y X + \mathrm{ad}_X Y|^2 - \langle \mathrm{ad}_X Y, \mathrm{ad}_X Y + \mathrm{ad}^\star{}_X Y\rangle - \langle \mathrm{ad}^\star{}_X X, \mathrm{ad}^\star{}_Y Y\rangle. \tag{19}$$

Remark 7. The advantage of writing the curvature in terms of the combination $(\mathrm{ad}^\star{}_X Y + \mathrm{ad}_X Y)$ is that this simplifies when X generates isometries of the Riemannian metric $\mathbf{g}$ defined by (11). Indeed, for any right-invariant fields Y and Z, the Lie derivative of the metric $\mathbf{g}$ is given by

$$0 = \mathcal{L}_X \mathbf{g}(Y, Z) = \langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = \langle [Y, X], Z\rangle + \langle [Z, X], Y\rangle \\ = \langle \mathrm{ad}_X Y, Z\rangle + \langle \mathrm{ad}_X Z, Y\rangle = \langle \mathrm{ad}_X Y + \mathrm{ad}^\star{}_X Y, Z\rangle.$$

So we see that X generates isometries of the metric iff $(\mathrm{ad}_X Y + \mathrm{ad}^\star{}_X Y) = 0$ for all Y. Thus if X is relatively simple as in our case, then the curvature should also be relatively simple to compute.

We now suppose $X = (f, \beta)$ and $Y = (g, \gamma)$, where $f = f(r)$ is a purely radial function, so that X is steady solution of (16) as in Corollary 6. The first term in the curvature formula (19) is the most complicated, so we will simplify it here; the second term is relatively easy, and the third term vanishes.

Lemma 8. *Suppose $X = (f, \beta)$ and $Y = (g, \gamma)$ with f a function of r alone. Set $u(r) = f'(r)/\varphi(r)$. Then the first term of the curvature formula* (19) *takes the form*

$$\operatorname{ad}^\star{}_X Y + \operatorname{ad}^\star{}_Y X + \operatorname{ad}_X Y = \big(\eta, 0\big), \qquad \textit{where} \tag{20}$$

$$(\alpha^2 - \Delta)\eta = 2u'(r)\,\frac{\partial^2 g}{\partial\theta\partial r} + \zeta(r)\,\frac{\partial g}{\partial\theta} \quad \textit{and} \quad \zeta(r) = \frac{2}{\varphi(r)}\frac{d^2}{dr^2}\big(\varphi(r)u(r)\big) - \beta - \alpha^2 u(r).$$

Here η satisfies the same boundary conditions as in Proposition 3.

Proof. We use the fact that $\operatorname{ad}_X Y = -[X, Y] = (-\{f, g\}, -\int_M \chi\{f, g\}\,dA)$ from Definition 2. As in the proof of Proposition 3, the Poisson bracket integral can be written

$$\int_M \chi\{f, g\}\,dA = \int_M \{\chi f, g\}\,dA - \int_M g\{f, \chi\}\,dA.$$

The first integral is zero as the integral of a Poisson bracket; the second is zero since $\{f, \chi\} = 0$, because f and χ are both functions of r alone. Hence $\operatorname{ad}_X Y = (-\{f, g\}, 0)$.

Let $\operatorname{ad}^\star{}_X Y + \operatorname{ad}^\star{}_Y X = \big(\rho, 0\big)$. From Proposition 3, we see that

$$\begin{aligned}(\alpha^2 - \Delta)\big(\rho - \{f, g\}\big) &= \{f, \alpha^2 g + \gamma\chi - \Delta g\} + \{g, \alpha^2 f + \beta\chi - \Delta f\} - (\alpha^2 - \Delta)\{f, g\} \\ &= -\{f, \Delta g\} + \beta\{g, \chi\} - \{g, \Delta f\} - \alpha^2\{f, g\} + \Delta\{f, g\},\end{aligned} \tag{21}$$

using the fact that $\{f, \chi\} = 0$ again, and the antisymmetry of the bracket.

We easily compute from the Poisson bracket formula (8) that

$$\{g, \chi\} = -\frac{\chi'(r)}{\varphi(r)}\,\frac{\partial g}{\partial\theta} = -\frac{\partial g}{\partial\theta},$$

using the assumption (9) on χ. Furthermore we have

$$\{f, g\} = \frac{f'(r)}{\varphi(r)}\,\frac{\partial g}{\partial\theta} = u(r)\,\frac{\partial g}{\partial\theta},$$

so that equation (21) with $\eta = \rho - \{f, g\}$ becomes

$$(\alpha^2 - \Delta)\eta = \Delta(ug_\theta) - u\Delta(g_\theta) - \beta g_\theta - \alpha^2 u g_\theta + \frac{(\Delta f)'}{\varphi}\,g_\theta. \tag{22}$$

Using the formulas $\Delta f = \frac{1}{\varphi}\frac{d}{dr}(\varphi f') = 2\varphi' u + \varphi u'$ and $\Delta(ug_\theta) - u\Delta g_\theta = (\Delta u)g_\theta + 2u' g_{r\theta}$, equation (22) simplifies to

$$(\alpha^2 - \Delta)\eta = \Big((\Delta u) - (\beta + \alpha^2 u) + \frac{1}{\varphi}\,\frac{d}{dr}(2\varphi' u + \varphi u')\Big) g_\theta + 2u' g_{r\theta},$$

which then simplifies to (20). $\square$

The second term in the curvature formula (19) simplifies substantially.

Lemma 9. *Suppose $X = (f, \beta)$ and $Y = (g, \gamma)$ with f a function of r alone, and g constant on the boundary (if any) of M. Set $u(r) = f'(r)/\varphi(r)$. Then the second term of the curvature formula* (19) *may be written as*

$$\langle \operatorname{ad}_X Y, \operatorname{ad}_X Y + \operatorname{ad}^\star{}_X Y\rangle = \int_M u'(r)^2 \Big(\frac{\partial g}{\partial\theta}\Big)^2\,dA.$$

Proof. Recall from the proof of Lemma 8 that $\operatorname{ad}_X Y = \big(-\{f, g\}, 0\big) = \big(-u(r)g_\theta, 0\big)$. Write $\operatorname{ad}^\star{}_X Y = (\phi, 0)$ where ϕ satisfies

$$(\alpha^2 - \Delta)\phi = \{f, \alpha^2 g + \gamma\chi - \Delta g\} = \alpha^2\{f, g\} - \{f, \Delta g\},$$

as in Proposition 3. Since f and g are both constant on the boundary of M, we know that $\{f,g\}$ vanishes on this boundary, and thus an integration by parts gives

$$\begin{aligned}
\langle \mathrm{ad}_X Y, \mathrm{ad}_X Y + \mathrm{ad}^\star{}_X Y\rangle &= \int_M \{f,g\}(\alpha^2-\Delta)\big(\{f,g\}-\phi\big)\,dA \\
&= \int_M \{f,g\}\Big(\alpha^2\{f,g\}-\Delta\{f,g\}-\alpha^2\{f,g\}+\{f,\Delta g\}\Big)\,dA \\
&= \int_M ug_\theta\Big(-\Delta(ug_\theta)+u\Delta(g_\theta)\Big)\,dA \\
&= \int_M ug_\theta\Big(-(\Delta u)g_\theta - 2u'g_{r\theta}\Big)\,dA \\
&= -\int_M u\Delta u(g_\theta)^2\,dA - \int_M uu'\frac{\partial}{\partial r}\big(g_\theta^2\big)\,dA.
\end{aligned}$$

We now integrate the second term by parts, using the fact that g_θ vanishes on boundary components, to get

$$\langle \mathrm{ad}_X Y, \mathrm{ad}_X Y + \mathrm{ad}^\star{}_X Y\rangle = \int_M \Big(-\frac{u}{\varphi}\frac{d}{dr}(\varphi u') + \frac{1}{\varphi}\frac{d}{dr}(\varphi u u')\Big) g_\theta^2\,dA = \int_M (u')^2 g_\theta^2\,dA.$$

□

As mentioned already, the last term in the curvature formula (19) vanishes since X is a steady solution of the Euler equation. The curvature thus takes the form

$$\langle R(X,Y)Y,X\rangle = \frac{1}{4}\int_M \Big(\alpha^2\eta^2 + |\nabla\eta|^2 - (u')^2 g_\theta^2\Big)\,dA, \tag{23}$$

where η satisfies (20). Heuristically (to highest order) we have $\eta_r \sim -2u'g_\theta$, which suggests there is another cancellation here. To see it, we need to solve the PDE (20), and the easiest way to do this is using a Fourier expansion.

Proposition 10. *Suppose $X=(f,\beta)$ and $Y=(g,\gamma)$, with f depending only on r and g expressed in a Fourier series as*

$$g(r,\theta) = \sum_{n\in\mathbb{Z}} g_n(r)e^{in\theta},$$

with boundary conditions $\varphi(r)g_n(r)=0$ at $r=0$ and $r=R$ for $n\neq 0$ (i.e., g is constant on the boundary, if any).

Let h_0 and h_1 be solutions of

$$\frac{1}{\varphi}\frac{d}{dr}\big(\varphi(r)h_i'(r)\big) - \Big(\alpha^2 + \frac{n^2}{\varphi(r)^2}\Big)h_i(r) = 0 \tag{24}$$

satisfying $h_0(0)=0$ and $h_1(R)=0$, and with Wronskian

$$\varphi(r)\big(h_1(r)h_0'(r) - h_0(r)h_1'(r)\big) = 1. \tag{25}$$

Define functions ξ_0 and ξ_1 by

$$\xi_i(r) = \Big(2\frac{d}{dr}(\varphi' u) - \varphi(\beta+\alpha^2 u)\Big)h_i(r) - 2\varphi(r)u'(r)h_i'(r). \tag{26}$$

Finally define

$$H_{0,n}(r) = \int_0^r \xi_0(s)g_n(s)\,ds \qquad \text{and} \qquad H_{1,n}(r) = \int_r^R \xi_1(s)g_n(s)\,ds. \tag{27}$$

Then the function η defined by Lemma 8 is given by

$$\eta(r,\theta) = \sum_{n\in\mathbb{Z}} \eta_n(r)e^{in\theta},$$

where the components satisfy

$$\eta_n(r) = in\big[h_1(r)H_{0,n}(r) + h_0(r)H_{1,n}(r)\big]. \tag{28}$$

Proof. The Wronskian is constant for two solutions of (24). If $\varphi(r) > 0$ for all $r \in [0, R]$, then we may obviously choose functions h_0 and h_1 unique up to a constant with $h_0(0) = 0$ and $h_1(R) = 0$ and scaled so that (25) is satisfied. If $\varphi(0) = 0$ then the condition $\varphi'(0) = 1$ implies, by the usual theory of ODEs at singular points, that $h_1(r) \sim r^{-n}$ and $h_0(r) \sim r^n$ as $r \to 0$, so that (25) can still be satisfied as $r \to 0$. Similarly if $\varphi(R) = 0$, we get the same asymptotics on the right side.

The equation (20) becomes, in Fourier components,

$$\left(\alpha^2 + \frac{n^2}{\varphi(r)^2}\right)\eta_n(r) - \frac{1}{\varphi(r)}\frac{d}{dr}\big(\varphi(r)\eta_n'(r)\big) = in\big(2u'(r)g_n'(r) + \zeta(r)g_n(r)\big), \tag{29}$$

with solution given for $n \neq 0$ by

$$\begin{aligned}\eta_n(r) = in\Big[h_1(r)\int_0^r h_0(s)\varphi(s)\big(2u'(s)g_n'(s) + \zeta(s)g_n(s)\big)\,ds \\ + h_0(r)\int_r^R h_1(s)\varphi(s)\big(2u'(s)g_n'(s) + \zeta(s)g_n(s)\big)\,ds\Big].\end{aligned}$$

We can easily check that this satisfies the boundary conditions $\eta_n(0) = \eta_n(R)$ whether φ is zero at the endpoints or not, and direct differentiation using (24) shows that it satisfies the differential equation as well.

Finally we integrate by parts to remove the $g_n'(s)$ term, using the fact that $\varphi(s)g_n(s)$ vanishes at $s = 0$ or $s = R$. We get

$$\begin{aligned}\eta_n(r) = in\Big[h_1(r)\int_0^r \big(h_0(s)\varphi(s)\zeta(s) - 2\frac{d}{ds}\big(h_0(s)\varphi(s)u'(s)\big)g_n(s)\,ds \\ + h_0(r)\int_r^R \Big(h_1(s)\varphi(s)\zeta(s) - 2\frac{d}{ds}\big(h_1(s)\varphi(s)u'(s)\big)\Big)g_n(s)\,ds\Big].\end{aligned} \tag{30}$$

Using the definition of ζ from (20), we get

$$\begin{aligned}&h_i(r)\varphi(r)\zeta(r) - 2\frac{d}{dr}\big(h_i(r)\varphi(r)u'(r)\big) \\ &\quad = h_i(r)\varphi(r)\left(\frac{2}{\varphi(r)}\frac{d^2}{dr^2}\big(\varphi(r)u(r)\big) - \beta - \alpha^2u(r)\right) - 2\frac{d}{dr}\big(h_i(r)\varphi(r)u'(r)\big) \\ &\quad = \left(2\frac{d}{dr}\big(\varphi'(r)u(r)\big) - \varphi(r)\big(\beta + \alpha^2u(r)\big)\right)h_i(r) - 2\varphi(r)u'(r)h_i'(r) \\ &\quad = \xi_i(r)\end{aligned} \tag{31}$$

by the definition (26) of ξ_i. Plugging into (30) gives (28) in case $n \neq 0$.

If $n = 0$, recall that the boundary conditions are different, and we require $\varphi(0)\eta_0'(0) = 0$ and $\varphi(R)\eta_0'(R) = 0$. But if $n = 0$, then the right side of (29) vanishes, and so the unique solution is $\eta_0(r) = 0$, which still fits (trivially) the formula (28). □

Combining the results of Lemmas 8 and 9, using the explicit solution for η from Proposition 10, the curvature formula (23) simplifies to the following form.

Theorem 11. *Suppose $X = (f, \beta)$ and $Y = (g, \gamma)$ with $f = f(r)$ and $g = \sum_{n\in\mathbb{Z}} g_n(r)e^{in\theta}$. Then the curvature from formula (23) is given by*

$$\langle R(X,Y)Y, X\rangle = \pi\sum_{n\in\mathbb{Z}} n^2 Re\int_0^R\int_0^r \xi_1(r)\xi_0(s)g_n(s)\overline{g_n(r)}\,ds\,dr, \tag{32}$$

where ξ_i is defined by formula (26).

Proof. Using Lemmas 8 and 9, the curvature formula (19) can be expressed in terms of the Fourier components as

$$\begin{aligned}\langle R(Y,X)X,Y\rangle = 2\pi \sum_{n\in\mathbb{Z}} \frac{1}{4}\Big(\int_0^R \Big(\big(\alpha^2 + \tfrac{n^2}{\varphi(r)^2}\big)|\eta_n(r)|^2 + |\eta_n'(r)|^2\Big)\varphi(r)\,dr \\ - n^2 \int_0^R u'(r)^2 |g_n(r)|^2 \varphi(r)\,dr\Big).\end{aligned} \tag{33}$$

Integrate the first term by parts, using the fact that if $n \neq 0$ we have $\eta_n(0) = \eta_n(R) = 0$, to get

$$\begin{aligned}&\int_0^R \Big(\big(\alpha^2 + \tfrac{n^2}{\varphi(r)^2}\big)|\eta_n(r)|^2 + |\eta_n'(r)|^2\Big)\varphi(r)\,dr \\ &\quad= \int_0^R \eta_n(r)\Big(\big(\alpha^2 + \tfrac{n^2}{\varphi(r)^2}\big)\overline{\eta_n(r)} - \frac{1}{\varphi(r)}\frac{d}{dr}\big(\varphi(r)\overline{\eta_n'(r)}\big)\Big)\varphi(r)\,dr \\ &\quad= -in \int_0^R \eta_n(r)\big(2u'(r)\overline{g_n'(r)} + \zeta(r)\overline{g_n(r)}\big)\varphi(r)\,dr,\end{aligned} \tag{34}$$

using (29).

Now we insert the solution (28) into (34) and integrate by parts to get

$$\begin{aligned}&- in \int_0^R \eta_n(r)\big(2u'(r)\overline{g_n'(r)} + \zeta(r)\overline{g_n(r)}\big)\varphi(r)\,dr \\ &\quad= -in \int_0^R \overline{g_n(r)}\Big(\big[\varphi(r)\zeta(r) - 2\tfrac{d}{dr}\big(\varphi(r)u'(r)\big)\big]\eta_n(r) - 2\varphi(r)u'(r)\big)\eta_n'(r)\Big)\,dr \\ &\quad= n^2 \int_0^R \overline{g_n(r)}\Big(\big[\varphi(r)\zeta(r) - 2\tfrac{d}{dr}\big(\varphi(r)u'(r)\big)\big]\big[h_1(r)H_{0,n}(r) + h_0(r)H_{1,n}(r)\big] \\ &\qquad\qquad - 2\varphi(r)u'(r)\eta_n'(r)\Big)\,dr,\end{aligned} \tag{35}$$

Formula (28) for η_n gives

$$\eta_n'(r) = in\big[h_1'(r)H_{0,n}(r) + h_0'(r)H_{1,n}(r) + h_1(r)H_{0,n}'(r) + h_0(r)H_{1,n}'(r)\big]. \tag{36}$$

The last two terms can be simplified using the definition (27) of H_0 and H_1 to get

$$\begin{aligned}h_1(r)H_{0,n}'(r) + h_0(r)H_{1,n}'(r) &= \big[h_1(r)\xi_0(r) - h_0(r)\xi_1(r)\big]g_n(r) \\ &= -2\varphi(r)u'(r)\big[h_1(r)h_0'(r) - h_0(r)h_1'(r)\big]g_n(r) \\ &= -2u'(r)g_n(r),\end{aligned} \tag{37}$$

using the Wronskian condition (25) in the last line. Inserting (37) into (36), then inserting that into (35) and using the formula (31) for ξ_i gives

$$\begin{aligned}&- in \int_0^R \eta_n(r)\big(2u'(r)\overline{g_n'(r)} + \zeta(r)\overline{g_n(r)}\big)\varphi(r)\,dr \\ &\quad= n^2 \int_0^R \overline{g_n(r)}\Big(\xi_1(r)H_{0,n}(r) + \xi_0(r)H_{1,n}(r)\big] + 4\varphi(r)u'(r)^2 g_n(r)\Big)\,dr.\end{aligned} \tag{38}$$

Finally inserting (38) into (33) gives the cancellation of the $|g_n(r)|^2$ term and the simplification

$$\begin{aligned}\langle R(Y,X)X,Y\rangle &= 2\pi \sum_{n\in\mathbb{Z}} \frac{1}{4}\int_0^R \overline{g_n(r)}\big[\xi_1(r)H_{0,n}(r) + \xi_0(r)H_{1,n}(r)\big]\,dr \\ &= \frac{\pi}{2}\sum_{n\in\mathbb{Z}}\int_0^R \big[H_{1,n}(r)\overline{H_{0,n}'(r)} - \overline{H_{1,n}'(r)}H_{0,n}(r)\big]\,dr.\end{aligned}$$

One final integration by parts, using the fact that $H_{1,n}(r)H_{0,n}(r)$ vanishes at both $r=0$ and $r=R$, establishes

$$\langle R(Y,X)X,Y\rangle = -\frac{\pi}{2}\sum_{n\in\mathbb{Z}}\int_0^R \big[H_{1,n}'(r)\overline{H_{0,n}(r)} + \overline{H_{1,n}'(r)}H_{0,n}(r)\big]\,dr,$$

which is (32). □

We have now obtained the most complete simplification possible of the original curvature formula (18) into the series (32) given by Theorem 11. Since all the Fourier components g_n may be chosen independently, the curvature $\langle R(Y,X)X,Y\rangle$ is nonnegative or nonpositive iff the same is true of every component of (32). The criterion for this to be true is given by the following theorem, which is of independent interest.

Theorem 12. *Suppose $\xi_0, \xi_1\colon [0,R]\to\mathbb{R}$ are given functions with only isolated zeroes in $(0,R)$, and define $\Psi(r) = \xi_1(r)/\xi_0(r)$. Then the bilinear form*

$$g \mapsto B(g,g) := 2\int_0^R\int_0^r \xi_1(r)\xi_0(s)\mathrm{Re}\big(\overline{g(s)}g(r)\big)\,ds\,dr \tag{39}$$

is nonpositive for all $g\colon[0,L]\to\mathbb{C}$ if and only if Ψ is nowhere zero or infinite in $(0,R)$, and the function Ψ is increasing and nonpositive on $[0,R]$.

Proof. Suppose $\Psi(r)$ is well-defined on $[0,L]$. Let $H_0(r) = \int_0^r \xi_0(s)g(s)\,ds$. Then we have

$$\begin{aligned}B(g,g) &= \int_0^R \Psi(r)\overline{H_0'(r)}H_0(r) + \Psi(r)H_0'(r)\overline{H_0(r)}\,dr \\ &= \int_0^R \Psi(r)\frac{d}{dr}|H_0(r)|^2\,dr = \Psi(R)|H_0(R)|^2 - \int_0^R \Psi'(r)|H_0(r)|^2\,dr.\end{aligned}\tag{40}$$

If $\Psi(R)\le 0$ and $\Psi'(r)\ge 0$, then $B(g,g)\le 0$ for every g.

Conversely, suppose that $B(g,g)\le 0$ for every g. We first claim that the function Ψ cannot have any zero in $(0,R)$. For suppose $\Psi(r_0)=0$ for some $r_0\in[a,b]\subset(0,R)$, and that Ψ is nonzero otherwise on $[a,b]$. The integral (40) can be written as

$$B(g,g) = \Psi(b)|H_0(b)|^2 - \int_a^b \Psi'(r)|H_0(r)|^2\,dr.$$

We know $\Psi(r_0)=0$ for a unique $r_0\in[a,b]$, and we consider the sign of $\Psi(b)$, since by assumption $\Psi(a)\neq 0$.

- If $\Psi(b)>0$ then we may clearly choose g so that $|H_0(b)|$ is large compared to $\|H_0\|_{L^2(a,b)}$ and obtain positivity of $B(g,g)$.
- If $\Psi(b)<0$ then Ψ' must be negative at some $c_0\in(r_0,b)$, and we may choose g so that H_0 is supported in a small neighborhood of c_0 and again obtain positivity of $B(g,g)$.

Thus if $B(g,g)\le 0$ for every g, then Ψ cannot have a zero in $(0,R)$.

Reversing the order of integration and defining $H_1(r) = \int_r^R \xi_1(s)g(s)\,ds$ allows us to write

$$\begin{aligned} B(g,g) &= 2\int_0^R \int_r^R \xi_1(s)\xi_0(r)\mathrm{Re}\big(\overline{g(r)}g(s)\big)\,ds\,dr \\ &= \int_0^R \int_r^R \xi_1(s)\xi_0(r)\big(\overline{g(r)}g(s) + \overline{g(s)}g(r)\big)\,ds\,dr \\ &= -\int_0^R \frac{1}{\Psi(r)}\frac{d}{dr}|H_1(r)|^2\,dr \\ &= \Psi^{-1}(0)\,|H_1(0)|^2 + \int_0^R (\Psi^{-1})'(r)|H_1(r)|^2\,dr. \end{aligned}$$

Hence the reciprocal of Ψ also cannot have a zero by the same reasoning. So Ψ must be well-defined in $(0,R)$.

Lastly, we claim that the function Ψ is increasing and nonpositive on $[0,R]$. If there is any point $r_0 \in (0,R)$ with $\Psi'(r_0) < 0$, in a small neighborhood of r_0 we can choose H_0 nonzero in this neighborhood and zero outside, and obtain a contradiction in the nonpositivity of (40). Hence we must have $\Psi'(r) \geq 0$ everywhere in $[0,R]$ by continuity. Meanwhile if $\Psi(L) > 0$, then we can choose H_0 such that $|H_0(R)|$ is large but $\|H_0\|_{L^2}$ is small on $[0,R]$, and again obtain a contradiction. This completes the proof of the converse. ☐

To help understand what the condition in Theorem 12 means when we try to make it more explicit, we write the definition (26) as

$$\xi_i(r) = \kappa(r)h_i(r) - 2\varphi(r)u'(r)h_i'(r), \qquad \kappa := 2\frac{d}{dr}(\varphi' u) - \varphi(\beta + \alpha^2 u) \tag{41}$$

We will need to ensure that neither ξ_0 nor ξ_1 is ever zero in the interior. Since the functions h_i and h_i' satisfy (24), with boundary conditions $h_0(0) = 0$ and $h_0'(0) \geq 0$ and $h_1(R) = 0$ and $h_1'(R) \leq 0$, they are both convex and nowhere zero in the interior. Hence everything must depend on the behavior of the two functions $\kappa(r)$ and $u'(r)$ in the interior. The following Corollary is helpful.

Corollary 13. *Suppose ξ_0 and ξ_1 are defined as in* (26) *and κ is defined by* (41)*. If the bilinear form* (39) *is nonnegative for all functions g, then $\frac{u'}{\kappa}$ has no zeroes on* $(0,R)$.

Proof. We want to show that $Z := \frac{u'}{\kappa}$ cannot approach zero; note that it is possible u' and κ are both zero at the same point, as long as the limit of the ratio is nonzero. It is also possible for κ to approach zero without u' approaching zero.

Suppose that $\lim_{r\to r_0} Z(r) = 0$ for some $r_0 \in (0,R)$. In a small neighborhood $(r_0-\varepsilon, r_0+\varepsilon)$, we may write

$$\frac{\xi_1(r)}{\xi_0(r)} = \frac{h_1(r) - 2Z(r)\varphi(r)h_1'(r)}{h_0(r) - 2Z(r)\varphi(r)h_0'(r)} = \frac{h_1(r)}{h_0(r)} + O(\varepsilon).$$

Consider functions g with support in $(r_0-\varepsilon, r_0+\varepsilon)$. Then formula (39) becomes, using the same integration by parts trick as in the proof of Theorem 12,

$$B(g,g) = \left[\frac{h_1(r_0+\varepsilon)}{h_0(r_0+\varepsilon)} + O(\varepsilon)\right]|H_0(r_0+\varepsilon)|^2 - \int_0^R \left[\frac{d}{dr}\left(\frac{h_1(r)}{h_0(r)}\right) + O(\varepsilon)\right]|H_0(r)|^2\,dr. \tag{42}$$

We know that $\frac{h_1}{h_0}$ is a positive function with negative derivative everywhere in $(0,R)$, and thus for sufficiently small ε both terms in (42) are positive, giving a contradiction. ☐

We are now ready to prove our main Theorem 1.

Proof of Theorem 1. By Corollary 13, the function $Q(r)$ given by (4) is well-defined on $(0, R)$. The ratio $\Psi(r) = \xi_1(r)/\xi_0(r)$ from Theorem 12, with ξ_i defined by (26) can be written in terms of Q as

$$\Psi(r) = \frac{Q(r)h_1(r) - \varphi(r)h_1'(r)}{Q(r)h_0(r) - \varphi(r)h_0'(r)}.$$

If $Q(r) > 0$, then the numerator of Ψ is positive while the denominator is negative if $Q(r) < \frac{\varphi(r)h_0'(r)}{h_0(r)}$. Similarly if $Q(r) < 0$, then the denominator is negative while the numerator is positive if $|Q(r)| > \frac{\varphi(r)|h_1'(r)|}{h_1(r)}$. We will investigate these inequalities in a moment, but clearly we need them satisfied in order for the condition $\Psi < 0$ to hold on $(0, R)$ as in Theorem 12.

The derivative of Ψ is

$$\begin{aligned}
\Psi' &= \frac{(Qh_0 - \varphi h_0')\big(Qh_1' + Q'h_1 - \frac{d}{dr}(\varphi h_1')\big) - (Qh_1 - \varphi h_1')\big(Qh_0' + Q'h_0 - \frac{d}{dr}(\varphi h_0')\big)}{(Qh_0 - \varphi h_0')^2} \\
&= \frac{(Qh_0 - \varphi h_0')\big(Qh_1' + Q'h_1 - \varphi(\alpha^2 + \frac{n^2}{\varphi^2})h_1\big) - (Qh_1 - \varphi h_1')\big(Qh_0' + Q'h_0 - \varphi(\alpha^2 + \frac{n^2}{\varphi^2})\big)}{(Qh_0 - \varphi h_0')^2} \\
&= \frac{(h_0h_1' - h_0'h_1)\big(Q^2 + \varphi Q' - (\alpha^2\varphi^2 + n^2)\big)}{(Qh_0 - \varphi h_0')^2} \\
&= -\frac{Q^2 + \varphi Q' - (\alpha^2\varphi^2 + n^2)}{\varphi(Qh_0 - \varphi h_0')^2},
\end{aligned}$$

using the formulas (24) for the ODE satisfied by h_i and (25) for the Wronskian simplification. We find that $\Psi' \geq 0$ iff

$$Q^2 + \varphi Q' - (\alpha^2\varphi^2 + n^2) \leq 0 \qquad \text{for all } r \in (0, R). \tag{43}$$

The condition (43) must be satisfied in order for the n^{th} term in the curvature formula given by Theorem 11 to be nonnegative. But all components g_n may be chosen independently, and thus for the entire expression to be nonpositive, every term must be. Thus (43) must be true for every nonzero integer n, and in particular it is true for $n = 1$, which is sufficient to satisfy it for all other n. (Note that when $n = 0$ the entire expression in (32) disappears anyway.) This yields the main inequality (5).

We now return to the issue of the sign of Q. If (43) is satisfied, then we want to show $Q(r) < P_0(r)$, where $P_0(r) = \frac{\varphi(r)h_0'(r)}{h_0(r)}$; this ensures that $\Psi < 0$ and is well-defined. Observe that P_0 satisfies the Riccati equation

$$\varphi(r)P_0'(r) + P_0(r)^2 = \alpha^2 + \frac{n^2}{\varphi^2(r)},$$

which is the same as (43) with equality instead of inequality. Thus we can write

$$\varphi \frac{d}{dr}(Q - P_0) + (Q - P_0)(Q + P_0) \leq 0,$$

which integrates to give the inequality

$$\big[P_0(r) - Q(r)\big] \geq \big[P_0(r_0) - Q(r_0)\big] \exp\left[-\left(\int_{r_0}^r \frac{Q(s) + P_0(s)}{\varphi(s)}\, ds\right)\right],$$

showing that if $Q < P_0$ is true anywhere, then it is true everywhere in $(0, R)$. Similarly we want to show that (43) implies $Q(r) < P_1(r)$, where $P_1(r) = \frac{\varphi(r)h_1'(r)}{h_1(r)}$. Clearly the same proof works. $\square$

4. Examples

Example 1 ($\varphi \equiv 1$). The most important example is the flat case on a rectangle, where $\varphi(r) \equiv 1$. In this case the formula (4) reduces to

$$Q(r) = -\frac{1}{2}\frac{\beta + \alpha^2 u(r)}{u'(r)},$$

and the condition (5) becomes

$$(6\alpha^2 + 4)u'(r)^2 - \big[\beta + \alpha^2 u(r)\big]^2 - 2\big[\beta + \alpha^2 u(r)\big]u''(r) \geq 0. \tag{44}$$

If both α and β are zero, the condition (44) reduces to $4u'(r)^2 \geq 0$ which is always satisfied; this reproduces the result known from [12], that every ideal shear velocity field $X = u(r)\,\partial_\theta$ has nonpositive curvature in all sections containing it. If $\alpha = 0$ and $\beta \neq 0$, the condition becomes

$$4u'(r)^2 - 2\beta u''(r) - \beta^2 \geq 0 \qquad \text{for all } r,$$

and for sufficiently large values of $|\beta|$ this is impossible; in other words large values of β stabilize the fluid in the sense of giving more directions in which curvature is positive (and thus Lagrangian perturbations remain bounded). The same thing happens in the case when $\beta = 0$ while $\alpha \neq 0$: the nonpositivity condition is

$$(6\alpha^2 + 4)u'(r)^2 - 2u(r)u''(r)\alpha^2 - u(r)^2\alpha^4 \geq 0,$$

which again becomes impossible to satisfy for sufficiently large α.

Writing $\beta + \alpha^2 u(r) = v(r)^k$ for $k = -\frac{\alpha^2}{2(\alpha^2+1)}$, and assuming $v(r) > 0$ for simplicity, the inequality (44) becomes

$$v''(r) \geq (\alpha^2 + 1)v(r).$$

Thus velocity fields that are "critical" for this inequality take the form

$$u(r) = \frac{1}{\alpha^2}\Big(c\cosh^k(ar + b)) - \beta\Big), \qquad a = \sqrt{\alpha^2 + 1}, \quad k = -\frac{\alpha^2}{2(\alpha^2 + 1)}, \quad b, c > 0. \tag{45}$$

Here we have chosen $c \geq 0$ so that $v(r) > 0$ on $[0, R]$, while $b > 0$ so that $u'(r) > 0$ on $[0, R]$. Graphs of some of these functions are shown in Figure 4.

Example 2 ($\varphi(r) = r$). Similarly on the disc where $\varphi(r) = r$, we would have $Q(r) \equiv 1$ in the case where $\alpha = \beta = 0$, so that (5) would automatically be satisfied. On the other hand for nonzero values of either α or β, the inequality becomes harder to satisfy and thus positive curvature is created. Specifically if we write $\beta + \alpha^2 u(r) = v(r)^{-1/2}$, then the inequality (5) becomes

$$3v'(r) + r\alpha^2 v(r) - rv''(r) \leq 0. \tag{46}$$

Critical functions (when (46) becomes equality) satisfy $v(r) = r^2 K_2(\alpha r)$ where K_2 is the modified Bessel function. We graph these for $\beta = 0$ and several values of α in Figure 4.

We note that despite the general sense that α and β being nonzero tends to create directions of positive sectional curvature, this is not always true. In some circumstances an appropriate choice of α and β can make a steady flow have nonpositive curvature even though the curvature would take both signs for $\alpha = \beta = 0$.

Example 3. Let $\varphi(r) = 1 + \frac{1}{2}r^2$ and $u(r) = r$, with $\beta = 0$ and $R = 1$. A picture is shown in Figure 4. Then

$$Q(r) = 2r - \frac{\alpha^2 r}{4}(r^2 + 2).$$

The inequality (5) becomes

$$r^2(r^2 + 2)^2\alpha^4 - (26r^4 + 64r^2 + 24)\alpha^2 + 80r^2 + 16 \leq 0. \tag{47}$$

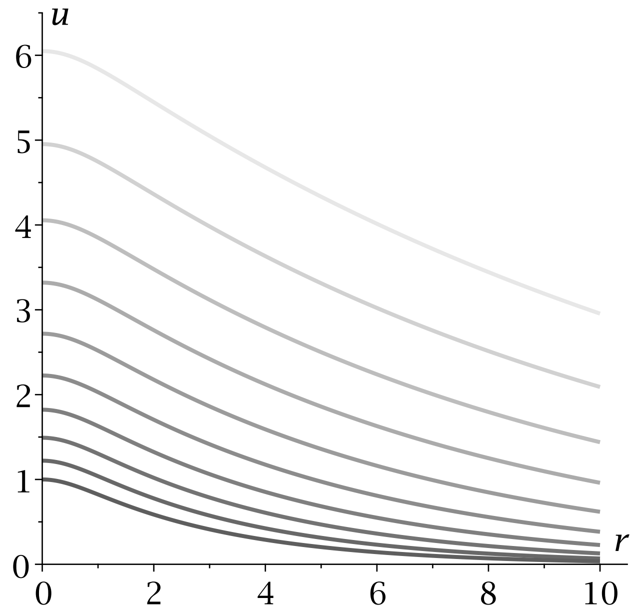

FIGURE 1. Graphs of the critical velocity profile (45) from Example 1 on the flat cylinder, for $\beta = 0$ and various values of $\alpha \in (0, 1)$. Lighter shades represent larger values of α.

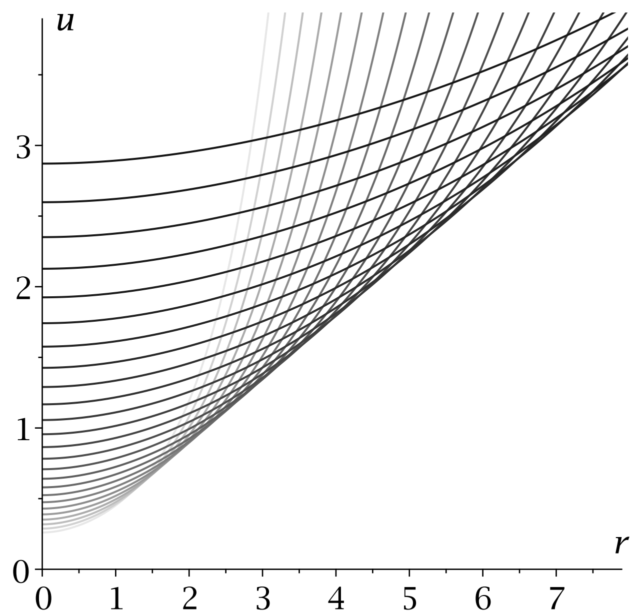

FIGURE 2. Critical velocity field profiles $u(r) = \dfrac{1}{\alpha^2 r\sqrt{K_2(\alpha r)}}$ on the disc with $\varphi(r) = r$ from Example 2, for values of α between 0 and 3. Lighter shades represent larger values of α.

If $\alpha = 0$ then (47) is obviously never satisfied on $[0, 1]$. Meanwhile if $\alpha = 1$ then (47) becomes

$$r^6 - 22r^4 + 20r^2 - 8 \leq 0,$$

which is true for $r \in [0, 1]$. In this case we get negative sectional curvature for all directions in the perturbed metric, while some directions have positive curvature in the unperturbed metric.

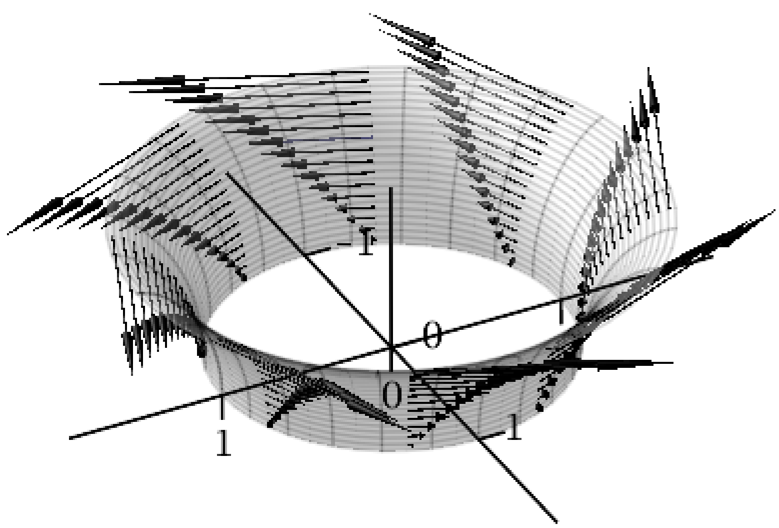

FIGURE 3. The velocity field $X = r\,\partial_\theta$ on a surface with $\varphi(r) = 1 + \frac{r^2}{2}$ from Example 3, represented as a surface of revolution in $\mathbb{R}^3$ parametrized as $\langle \varphi(r)\cos\theta, \varphi(r)\sin\theta, g(r)\rangle$ for $g(r) = \frac{1}{2}\big(\arcsin r + r\sqrt{1-r^2}\big)$. This velocity field has stable perturbations when $\alpha = 0$, but all curvatures are negative along it when $\alpha = 1$.

Example 4 ($\varphi(r) = \sin r$). In the case of a sphere with $u(r) = \cos(r)$ and $R = \pi$, first consider the case when $\alpha = \beta = 0$. From the result of [12], there is no steady flow that satisfy nonpositive criterion (5) on the entire sphere. In fact, we have $Q(r) = 2\cos(r)$ and the condition (5) becomes $3\cos(2r) \leq 0$, which is not satisfied on $[0, \pi]$. If we set α or β nonzero, the condition (5) becomes

$$6(2+\alpha^2)u'(r)^2 + 6\cot r\,\big(\beta + (2+\alpha^2)u(r)\big)\,u'(r) - \big(\beta + (2+\alpha^2)u(r)\big)^2 \\ - 2\,\big(\beta + (2+\alpha^2)u(r)\big)\,u''(r) \geq 0$$

Again, this inequality is impossible to satisfy when $|\alpha|$ or $|\beta|$ are sufficiently large. Thus, in this case it appears that nonzero values of α or β have a stabilizing effect on the Lagrangian perturbations of the quasi-geostrophic flow.

Acknowledgement. J. Lee was supported by Gustafsson Foundation, Sweden and the European Research Council, Grant Agreement No. 682537 and S. C. Preston was supported by Simons Foundation, Collaboration Grant for Mathematicians, no. 318969.

References

[1] Arnold, V.I., On the differential geometry of infinite-dimensional Lie groups and its application to the hydrodynamics of perfect fluids, translation of the French original, in *Vladimir I. Arnold: collected works vol. 2*, Springer, New York, 2014.

[2] V. Arnold and B. Khesin, Topological methods in hydrodynamics, *Springer*, Vol. 125, 1999

[3] M. Bauer, P. Harms, and S.C. Preston, Vanishing distance phenomena and the geometric approach to SQG, arXiv:1805.04401

[4] D. Ebin and J. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, *Ann. Math.*, 102–163, 1970

[5] D. Ebin and S. Preston, Riemannian geometry of the contactomorphism group, *Arnold Math. J.*, 1(1), 5–36, 2015

[6] D. Holm and V. Zeitlin, Hamilton's principle for quasigeostrophic motion, *Phys. Fluids*, 10(4), 800–806, 1998

[7] B. Khesin, J. Lenells, G. Misiolek, and S. Preston, Curvatures of Sobolev Metrics on Diffeomorphism Groups *Pure Appl. Math. Quart.*, 9(2), 291–332, 2013

[8] A. Majda, Introduction to PDEs and Waves for the Atmosphere and Ocean, *Amer. Math. Soc.*, Vol. 9, 2003

[9] G. Misiolek, Stability of Flows of Ideal Fluids and the Geometry of the Group of Diffeomorphisms, *Indiana Univ. Math. J.*, 42(1), 215–235, 1993

[10] J. Pedlosky, Geophysical fluid dynamics, *Springer*, 2013.

[11] S.C. Preston, For ideal fluids, Eulerian and Lagrangian instabilities are equivalent, *Geom. Funct. Anal.* 14(5), 1044–1062, 2004

[12] S.C. Preston, Nonpositive curvature on the area-preserving diffeomorphism group, *J. Geom. Phys.*, 53(2), 226–248, 2005

[13] C. Vizman, Cocycles and stream functions in quasigeostrophic motion, *J. Nonlin. Math. Phys.*, 15(2), 140–146, 2008

[14] P. Washabaugh, The SQG equation as a geodesic equation, *Arch. Rat. Mech. Anal.*, 222(3), 1269–1284, 2016

[15] P. Washabaugh and S.C. Preston, The geometry of axisymmetric ideal fluid flows with swirl, *Arnold Math. J.* 3(2), 175—185, 2017

[16] V. Zeitlin and R.A. Pasmanter, On the differential geometry approach to geophysical flows, *Phys. Lett. A*, 189, 59–63, 1994