FREDHOLM PROPERTIES OF RIEMANNIAN EXPONENTIAL MAPS ON DIFFEOMORPHISM GROUPS

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ABSTRACT. We prove that exponential maps of right-invariant Sobolev H^r metrics on a variety of diffeomorphism groups of compact manifolds are nonlinear Fredholm maps of index zero as long as r is sufficiently large. This generalizes the result of [EMP] for the L^2 metric on the group of volume-preserving diffeomorphisms important in hydrodynamics. In particular, our results apply to many other equations of interest in mathematical physics. We also prove an infinite-dimensional Morse Index Theorem, settling a question raised by Arnold and Khesin [AK] on stable perturbations of flows in hydrodynamics. Finally, we include some applications to the global geometry of diffeomorphism groups.

CONTENTS

1. INTRODUCTION

Several well-known nonlinear partial differential equations of mathematical physics arise as geodesic equations on various infinite-dimensional Lie groups. The first and perhaps the most fundamental example are the Euler equations of ideal hydrodynamics. In his celebrated paper [Ar1] Arnold showed that fluid motions correspond to geodesics in the group of volume-preserving diffeomorphisms (volumorphisms) equipped with a right-invariant metric defined by the kinetic energy of the fluid, a result which can

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be used formally to describe curvature and hence stability of fluid motions. Subsequently, Ebin and Marsden [EM] showed that the volumorphism group can be given the structure of a Hilbert manifold and proved that the corresponding geodesic equation is actually a smooth ordinary differential equation, hence rigorously justifying the infinite-dimensional geometric constructions.

Since then, much research has focused on finding such geometric formulations for other conservative systems of continuum mechanics, showing that the corresponding PDE admit smooth solutions that are critical points of some energy functional on an infinite-dimensional Lie group with a right-invariant metric. Once this is known, it is then tempting to use geometric intuition in order to better understand the qualitative behavior of solutions; for example, showing that the curvature is negative or positive can hopefully demonstrate existence of unstable or stable perturbations of the motion, respectively. However, this method is most powerful when, as in hydrodynamics, the equation is transformed from a *partial* differential equation on the manifold into a genuine *ordinary* differential equation on the diffeomorphism group, so that unique solutions can be constructed by standard techniques of ODE theory (rather than the use of ad hoc PDE estimates). One can then proceed to deduce rigorous results about the solutions from the study of the attendant geometric objects such as the exponential map, sectional curvature, the index form, conjugate points, etc.

After finding that the curvature of the hydrodynamical volumorphism group was often negative but sometimes positive, Arnold [Ar1] (see also [Ar2] and [AK]) asked about the existence and nature of conjugate points on the volumorphism group, particularly whether conjugate point locations could accumulate along geodesics. Such phenomena can happen in infinite-dimensional geometry generally, but if the exponential map happens to be a nonlinear Fredholm map, this possibility is precluded.

The first result on Fredholmness of a Riemannian exponential map was proved in [Mi2] under the assumption that the curvature operator along geodesics is compact. This holds for example on the free loop space, but it seems too restrictive for many other examples of interest. In a recent paper [EMP] we showed that the exponential map is Fredholm for the L^2 metric on the volumorphism group in two dimensions but not in three dimensions, using the algebraic structure rather than the curvature.

One of the goals of the present work is to extend the main result of [EMP] to exponential maps on diffeomorphism groups induced by other right-invariant metrics of interest in mathematical physics. Another goal is to provide a general framework within which one can study the associated nonlinear PDE using methods of infinite-dimensional Riemannian geometry.

Our main examples will be obtained in the following way. Given a compact Riemannian manifold M of dimension n, let $\mathcal{D}^s(M)$ be the group of diffeomorphisms with the Sobolev H^s topology, where $s > n/2+1$. It is well known that $\mathcal{D}^s(M)$ is a smooth Hilbert manifold as well as a topological group. We will be mainly interested in $\mathcal{D}^s(M)$ and the volumorphism group $\mathcal{D}^s_\mu(M)$, as well as central extensions of the diffeomorphism group. All such objects will be denoted by G^s. For each index $r \le s$ we will consider a Riemannian metric on $\mathcal{D}^s(M)$ defined at the identity by a Sobolev H^r inner product. We then right-translate this inner product to other tangent spaces to obtain a right-invariant metric on the group.

If $r = s$ we get a *strong Riemannian* metric (i.e., one whose distance function generates the underlying topology of the manifold), while if $r < s$ we get a *weak Riemannian* metric. Strong metrics have nice global geometric properties, while weak metrics tend to arise naturally in applications. The metric on a central extension is given by a direct product, which again is the case arising in applications.

This construction is sufficiently general to include many of the well-known PDE of mathematical physics, such as: the Euler equations of ideal hydrodynamics ($G^s = \mathcal{D}^s_\mu(M)$ with $r = 0$); the Lagrangian-averaged Euler equation ($G^s = \mathcal{D}^s_\mu(M)$ with $r = 1$); Burgers' equation ($G^s = \mathcal{D}^s(S^1)$ with $r = 0$); the Camassa-Holm equation ($G^s = \mathcal{D}^s(S^1)$ with $r = 1$); the EPDiff equation ($G^s = \mathcal{D}^s(M)$ with $r = 1$); and the KdV equation ($G^s = \mathrm{Vir}(S^1)$, the Bott-Virasoro group, with $r = 0$). Of course there are many other groups whose right-invariant metrics lead to interesting PDEs; for examples of such equations see e.g., Arnold and Khesin [AK], Marsden and Ratiu [MR], Schmid [Sch], Taylor [T2], Vizman [V3] or the recent book by Khesin and Wendt [KW] and their references.

Our main results deal with existence of Fredholm exponential maps of the right-invariant H^r-metrics on G^s and their properties. Specifically, after reviewing necessary background material in Section 2, we derive explicit formulas for the (Lie group and Lie algebra) coadjoint operators of the H^r metrics and write down the corresponding Euler equations in Section 3. Next, we show that if r is a sufficiently large integer, then the geodesic equation of the H^r right-invariant metric is locally well-posed for any of the groups G^s, with solutions depending differentiably on the initial data for any sufficiently large s (depending on r). In particular, we obtain a smooth exponential map $\exp_e$ defined on some neighborhood of zero in T_eG^s (Section 4) and thus can study its singularities as conjugate points. This is done in Section 5 where we also present examples showing that in infinite dimensions distribution of conjugate points can be very complicated.

Sections 6 and 7 contain the main constructions of this paper. We show that for each G^s, there is a critical index r_o such that if r is an integer with $r > r_o$, the differential of the H^r exponential map $d\exp_e(v)\colon T_eG^s \to T_{\exp_e(v)}G^s$ is a Fredholm operator of index zero, i.e., a bounded linear map with closed range, such that the kernel and cokernel have finite and equal dimensions. Thus, $\exp_e$ is a smooth Fredholm map of index zero in the sense of Smale [Sma] and, as a result, all conjugate points of the H^r metric that appear when $r > r_o$ must be necessarily of finite order and isolated along finite geodesic segments. The values of r_o required are

- If $G^s = \mathcal{D}^s(M)$, then $r_o = \frac{1}{2}$.
- If $G^s = \mathcal{D}^s_\mu(M)$, then $r_o = -\frac{1}{2}$ if $\dim(M) = 2$ and $r_o = 0$ otherwise.
- If $G^s = \mathrm{Vir}^s(S^1)$, then $r_o = \frac{3}{2}$.

In particular, this includes the result of [EMP] for the L^2 metric in 2D hydrodynamics and implies that the H^1 metrics which generate the Camassa-Holm equation and the Lagrangian-averaged Euler equation (in 2D as well as 3D) all have Fredholm exponential maps. Furthermore, it shows that failure of Fredholmness is a *borderline* case in 3D hydrodynamics, while it is *not* borderline for Burgers' equation or the Korteweg-de Vries equation.

We point out that the exponential map in an infinite-dimensional Riemannian manifold will typically not be Fredholm. Grossman [Gr] gave the first examples of this: on a sphere in a Hilbert space, the exponential map differential may have infinite-dimensional kernel (corresponding to an infinite-dimensional family of geodesics joining two antipodal points); in addition on an infinite-dimensional ellipsoid, the exponential map differential may fail to be surjective even if it is injective (which arises from a convergent sequence of conjugate point locations along a geodesic segment). For more explicit details on the pathological nature of conjugate points in infinite-dimensional manifolds we refer the reader to the recent papers of Biliotti, Exel, Piccione and Tausk [BEPT] for strong metrics, [P3] for a weak metric on the volumorphism group of a three-dimensional manifold, or Kappeler, Loubet and Topalov [KLT] for another weak metric on the full diffeomorphism group of the flat two-torus. We emphasize again that such phenomena cannot appear when the exponential map is Fredholm.

As mentioned above, Fredholmness comes in an essential way from the group structure and in particular the decomposition of the Jacobi equation into decoupled first-order equations, rather than from convenient properties of the curvature as on the free loop space [Mi2]. In Section 6 we will also show that for the H^1 metric on $\mathcal{D}^s(S^1)$ and for the L^2 metric on $\mathcal{D}^s_\mu(M^2)$, the curvature operator is not compact; in fact in both situations there is an infinite-dimensional subspace on which the curvature operator is positive and bounded away from zero. However in both examples the exponential map is Fredholm.

In the last two sections of the paper we describe two applications of our Fredholmness results. First, in Section 8, we prove the Morse Index theorem for geodesics of the L^2 metric on $\mathcal{D}^s_\mu(M^2)$ thus settling a question raised by Arnold and Khesin (see [AK], Chapter 4, p. 225). Finally, in Section 9, we describe surjectivity properties of the exponential map in two cases of particular interest: that of a strong Riemannian metric on G^s and the L^2 metric on $\mathcal{D}^s_\mu(M^2)$. The latter can be viewed as a step toward answering a conjecture of Shnirelman [Shn3].

2. Background

To deal with diffeomorphism groups as manifolds, one must specify a function space topology. At one extreme, it can be convenient to work with only C^∞ diffeomorphisms, although this requires a Fréchet topology in which important results like the implicit function theorem are not valid. At the other extreme, one can work in the topology generated by the kinetic energy of a physical system, although the diffeomorphisms may not form a smooth or even a C^0 manifold in such a topology. In either case one must often prove individual results using methods unique to a particular equation, rather than using general techniques of infinite-dimensional manifolds. Hence it has been common to work with intermediate Sobolev topologies, which are useful in a wide array of situations; we will use this approach, following Ebin and Marsden [EM].

For an n-dimensional manifold M, the class $H^s(M, M)$ is defined as the set of maps $\eta\colon M \to M$ which are of Sobolev class H^s in every coordinate chart. If $s > n/2+1$, then every $\eta \in H^s(M, M)$ is also C^1 by the Sobolev embedding theorem, and we can define $\mathcal{D}^s(M)$ as the open subset of $H^s(M, M)$ such that η^{-1} exists and is also in $H^s(M, M)$. Then $\mathcal{D}^s(M)$ is a topological group, where right translation $R_\eta\colon \xi \mapsto \xi \circ \eta$ is C^∞ and left translation $L_\eta\colon \xi \mapsto \eta \circ \xi$ is continuous (but not even Lipschitz continuous) in the H^s topology.

Given any Riemannian metric on M, we can define an H^s Riemannian metric on $\mathcal{D}^s(M)$ using the powers Δ^s of the Laplace-de Rham operator on M. However, for the most part, the geometry of this metric is not interesting from a physical point of view; in applications, the interesting metrics are *weak*, in that they do not generate the topology of the underlying space. Weak metrics give rise to many of the partial differential equations of physics, but the diffeomorphism group is often not a manifold in the topology generated by the weak metric. This is the source of many of the complications arising in infinite-dimensional geometry. We will have to deal with some of them in later sections.

We proceed to recall some basic facts about the structure of the various diffeomorphism groups and central extensions that will be studied here. In particular, we will need formulas for the group as well as the Lie algebra adjoint representations. For concreteness, we consider three situations: the full diffeomorphism group $\mathcal{D}^s(M)$, the volumorphism group $\mathcal{D}^s_\mu(M)$, and the Bott-Virasoro group $\mathrm{Vir}^s(S^1)$.

Definition 2.1. The *volumorphism group* $\mathcal{D}_\mu(M)$ is defined in terms of a Riemannian volume form μ on a compact n-dimensional Riemannian manifold M without boundary. It consists of those diffeomorphisms η such that $\eta^*\mu = \mu$. If $s > n/2+1$ then the group of volumorphisms of Sobolev class H^s of M,

$$\mathcal{D}^s_\mu(M) = \{\eta \in \mathcal{D}^s(M) \,|\, \eta^*\mu = \mu\},$$

is a smooth submanifold of the Hilbert manifold $\mathcal{D}^s(M)$. Its tangent space at the identity diffeomorphism consists of H^s divergence-free vector fields

$$T_e\mathcal{D}^s_\mu(M) = \{u \in T_e\mathcal{D}^s(M) \,|\, \operatorname{div} u = 0\}.$$

Definition 2.2. We define the *exact volumorphism group* $\mathcal{D}^s_{\mu,\mathrm{ex}}(M)$ in the following way. We recall the musical isomorphisms $\sharp$ and $\flat$: if α is a 1-form, then $\alpha^\sharp$ is the vector field such that $\alpha(v) = \langle \alpha^\sharp, v\rangle$ for every vector field v; $\flat$ is its inverse. We have the operator d which takes k-forms of class H^s to $(k+1)$-forms of class H^{s-1}, and its formal dual δ which takes $(k+1)$-forms of class H^s to k-forms of class H^{s-1}, defined so that for any k-form α and any $(n-k-1)$-form β, we have

$$\int_M \langle d\alpha, \beta\rangle \, d\mu = \int_M \langle \alpha, \delta\beta\rangle \, d\mu.$$

See for example [EM]. We also have the Hodge star operator $\star$ which maps k-forms to $(n-k)$-forms and is defined by $\langle \alpha, \beta\rangle \mu = \alpha \wedge \star\beta$, for any k-forms α and β. The relationship between the two on k-forms is

$$\delta = (-1)^{nk-n-1} \star d \star.$$

We rewrite the divergence as an operator on 1-forms, $\operatorname{div} u = -\delta u^\flat$, and use the Hodge decomposition [EM] to say that any H^s divergence-free vector field u may be written as $u^\flat = \delta\beta + h$, where β is an H^{s+1} 2-form and h is a C^∞ harmonic 1-form (i.e., one satisfying $\delta h = 0$ and $dh = 0$). The *exact volumorphism Lie algebra* is the set of those u such that $h = 0$, and the *exact volumorphism group* is the image of these under the Lie exponential map. (That it is actually a Lie algebra will follow from Remark 2.6.) Of course if the first homology $H^1(M) = 0$, the exact volumorphism group coincides with the volumorphism group.

In two dimensions, a 2-form β may be identified with a function $f = \star\beta$, and we write $u = \operatorname{sgrad} f \equiv (\delta\star f)^\sharp$, so that on the flat torus $\operatorname{sgrad} f = f_y \, \partial_x - f_x \, \partial_y$. In three dimensions, a 2-form β may be identified with a 1-form $\alpha = \star\beta$ and thus with a vector field $w = \alpha^\sharp$, and we write $u = \operatorname{curl} w \equiv (\delta\star w^\flat)^\sharp$.

Definition 2.3. Given any diffeomorphism group, we define a *central extension* in the following way: we find a 2-cocycle $B\colon \mathcal{D}^s(M) \times \mathcal{D}^s(M) \to \mathbb{R}^m$ satisfying the property

$$B(\eta, \xi) + B(\eta\circ\xi, \chi) = B(\xi, \chi) + B(\eta, \xi\circ\chi) \text{ for all } \eta, \xi, \chi \in \mathcal{D}^s(M). \tag{2.1}$$

(The existence of a nontrivial 2-cocycle depends on the cohomology of the diffeomorphism group.) The central extension of the diffeomorphism group is then defined as $\mathcal{D}^s(M) \times \mathbb{R}^m$ with group law

$$(\eta, \rho)\cdot(\xi, \sigma) = (\eta\circ\xi, \rho + \sigma + B(\eta, \xi)), \tag{2.2}$$

and equation (2.1) ensures that this group operation is associative.

The best known example of a central extension is the Bott-Virasoro group $\mathrm{Vir}(S^1)$.

Definition 2.4. The *Bott-Virasoro group* $\mathrm{Vir}^s(S^1)$ is the universal central extension of the group of orientation preserving H^s diffeomorphisms of the circle. Here B is the Bott cocycle

$$B(\eta, \xi) = \frac{1}{2}\int_{S^1} \log \partial_x(\eta\circ\xi) \, d\log \partial_x \xi. \tag{2.3}$$

The Bott-Virasoro group is known to be the configuration space of the Korteweg-de Vries [OK] and the Camassa-Holm equations [Mi3], both of which arise as water-wave equations.

For any group G^s we have two adjoint operators defined on the Lie algebra $\mathfrak{g}^s = T_eG^s$: the group adjoint $\mathrm{Ad}_\eta \colon \mathfrak{g}^s \to \mathfrak{g}^s$ defined by

$$\mathrm{Ad}_\eta v = dL_\eta dR_{\eta^{-1}} v, \tag{2.4}$$

and the Lie algebra adjoint $\mathrm{ad}_u v \colon \mathfrak{g}^s \to \mathfrak{g}^s$ defined by

$$\mathrm{ad}_u v = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\eta(t)} v, \tag{2.5}$$

where $\eta(t)$ is any curve in G^s with $\dot\eta(0) = u$. The following proposition gives the formulas for these objects on the groups we have been considering.

Proposition 2.5. *Suppose G^s is the diffeomorphism group, the exact volumorphism group, or the Bott-Virasoro group. Then the Lie group adjoint* (2.4) *and Lie algebra adjoint* (2.5) *are given by the following formulas:*

- *If G^s is the diffeomorphism group $\mathcal{D}^s(M)$ with Lie algebra $\mathfrak{g}^s$ of vector fields, then the group adjoint $\mathrm{Ad}_\eta \colon \mathfrak{g}^s \to \mathfrak{g}^s$ is given by*

$$\mathrm{Ad}_\eta v = \eta_* v = (D\eta\, v) \circ \eta^{-1}, \tag{2.6}$$

  while the Lie algebra adjoint $\mathrm{ad}_u \colon \mathfrak{g}^s \to \mathfrak{g}^s$ is given by

$$\mathrm{ad}_u v = -[u, v], \tag{2.7}$$

  the negative of the standard Lie bracket of vector fields.
- *If G^s is the exact volumorphism group as in Definition 2.2 with Lie algebra $\mathfrak{g}^s$ of vector fields of the form $v = (\delta\beta)^\sharp$, then the adjoints above can be written in the simplified form*

$$\begin{cases} \mathrm{Ad}_\eta \operatorname{sgrad} g = \operatorname{sgrad}(g \circ \eta^{-1}) & \textit{if } \dim(M) = 2, \\ \mathrm{Ad}_\eta \operatorname{curl} w = \operatorname{curl}\big((\eta^{-1})^* w^\flat)\big)^\sharp & \textit{if } \dim(M) = 3, \\ \mathrm{Ad}_\eta (\delta\beta)^\sharp = \big(\delta\big[\star(\eta^{-1})^*\star\beta\big]\big)^\sharp & \textit{in general.} \end{cases} \tag{2.8}$$

  The Lie algebra adjoint can be written in the form

$$\begin{cases} \mathrm{ad}_u v = \operatorname{sgrad}(u \times v) & \textit{if } \dim(M) = 2, \\ \mathrm{ad}_u v = \operatorname{curl}(u \times v) & \textit{if } \dim(M) = 3, \\ \mathrm{ad}_u v = (\delta(u^\flat \wedge v^\flat))^\sharp & \textit{in general.} \end{cases} \tag{2.9}$$

- *If G^s is $Vir^s(S^1)$, then the group adjoint action is given by*

$$\mathrm{Ad}_{(\eta,\rho)}(v, q) = \left(\eta_* v, q + \int_{S^1} S_\eta(\theta) v(\theta)\, d\theta\right), \tag{2.10}$$

  where

$$S_\eta = \frac{\eta'''}{\eta'} - \frac{3}{2}\Big(\frac{\eta''}{\eta'}\Big)^2 \tag{2.11}$$

is the Schwarzian derivative of η. The Lie algebra adjoint action is given by

$$\mathrm{ad}_{(u,p)}(v,q) = \left(-[u,v], \int_{S^1} u'''(\theta)v(\theta)\,d\theta\right). \tag{2.12}$$

Proof. These computations are standard; see for example [AK] or [KW]. The reason for the minus sign in (2.7) is the fact that the standard Lie derivative comes from $\mathrm{Ad}_{\eta^{-1}}$:

$$\mathcal{L}_u v = [u,v] = \frac{d}{dt}\Big|_{t=0} \eta(t)_*^{-1} v = \frac{d}{dt}\,\mathrm{Ad}_{\eta(t)^{-1}}\,v = -\,\mathrm{ad}_u\,v.$$

We will focus on the exact volumorphism group, where the computations are less known. So let α be a 1-form, β be a 2-form, and η be a volumorphism. If $v = (\delta\beta)^\sharp$, then

$$\alpha(\eta_* v)\circ\eta = \eta^*\alpha(v) = \langle \eta^*\alpha, v^\flat\rangle = \langle \eta^*\alpha, \delta\beta\rangle.$$

Therefore since η is volume-preserving,

$$\begin{aligned}\int_M \alpha(\eta_* v)\,d\mu &= \int_M \alpha(\eta_* v)\circ\eta\,d\mu = \int_M \langle \eta^*\alpha, \delta\beta\rangle\,d\mu = \int_M \langle d\eta^*\alpha, \beta\rangle\,d\mu \\ &= \int_M \eta^* d\alpha \wedge \star\beta = \int_M d\alpha \wedge (\eta^{-1})^*\star\beta \\ &= \int_M d\alpha \wedge \star\star(\eta^{-1})^*\star\beta = \int_M \langle \alpha, \delta\star(\eta^{-1})^*\star\beta\rangle\,d\mu.\end{aligned}$$

Since this is true for every 1-form α, we obtain the general case of (2.8). The special case when $n = 2$ follows from the fact that if f is a function, then $\eta^* f = f\circ\eta$. The formula for the Lie algebra adjoint follows from the general formula

$$[u,v]^\flat = -\delta(u^\flat \wedge v^\flat) + (\mathrm{div}\,v)u^\flat - (\mathrm{div}\,u)v^\flat.$$

□

Remark 2.6. Observe from formula (2.9) that on the volumorphism group, $\mathrm{ad}_u\,v$ is always an element of the exact volumorphism Lie algebra from Definition 2.2. Hence in particular the exact volumorphism Lie algebra is a Lie subalgebra.

On the diffeomorphism group $\mathcal{D}^s(M)$ of a compact manifold without boundary, we put a right-invariant H^r metric defined at the identity by

$$\langle u,v\rangle_{H^r} = \int_M \langle u, A^r v\rangle\,d\mu \tag{2.13}$$

for any $u, v \in T_e\mathcal{D}^s(M)$, where $A^r \in OPS^{2r}_{1,0}$ is an elliptic invertible operator of order $2r$. (For this to be usable, we may need to require s to be larger; for example, $s > n/2 + 1 + 2r$ will be enough so that if $v \in T_e\mathcal{D}^s$, then $A^r v$ is C^1.) Typical examples include $A^r = (\mathrm{id} + \Delta^r)$ or $A^r = (\mathrm{id}+\Delta)^r$, where $\Delta v = (d\delta v^\flat + \delta d v^\flat)^\sharp$ is the positive-definite Hodge Laplacian. Alternatively, if r is a positive integer, we can use $A^r = \sum_{k=0}^r \Delta^k$ again using the positive-definite Laplacian. In case M is the circle S^1, this becomes $A^r = \sum_{k=0}^r (-1)^k \partial_x^{2k}$ (as in [CKKT]).

Remark 2.7. In general we need not assume that r is an integer. Generally, any self-adjoint first order elliptic operator A on a compact manifold gives rise to a pseudo-differential operator A^z in $OPS^{Re(z)}_{1,0}$ for any complex number $z \in \mathbb{C}$ by the spectral theorem. However, all known examples which lead to physical differential equations involve $r = 0$ or $r = 1$, and the proof of local existence in Section 4 simplifies greatly if r is an integer. Hence we will assume that r is an integer as needed.

If A^r commutes with the operators d and δ, as we will always assume, then the right-invariant H^r metric (2.13) restricts to an H^r metric on the exact volumorphism group. At the identity it is given on 2-forms α and β by

$$\langle (\delta\alpha)^\sharp, (\delta\beta)^\sharp \rangle_{H^r} = \langle A^r \Delta\alpha, \beta \rangle_{L^2}, \tag{2.14}$$

where $\Delta = \delta d + d\delta$. Hence in terms of the "potentials" α and β, we get an H^{r+1} metric; however, we will continue to refer to it as the H^r metric on the actual vector fields.

The right-invariant metric on the Bott-Virasoro group is simply the Cartesian product:

$$\langle (u, p), (v, q) \rangle = \langle u, v \rangle_{H^r} + pq. \tag{2.15}$$

There are a number of other situations amenable to the type of analysis we consider here, although to keep the formulas relatively simple, we are avoiding this full generality. One can consider other subgroups of the diffeomorphism group, such as those preserving a symplectic form or a contact form; the geometry of the L^2 metric on these groups was considered by Smolentsev [Smo2]. One can also consider a semidirect product of a diffeomorphism group with a space of functions, forms, or vector fields. See Vizman [V3] for a review of equations arising through L^2 geometry on such groups. Several other examples are collected in the recent book [KW].

3. Coadjoint representation and Euler equations

The Lie group adjoint and Lie algebra adjoint formulas appearing in Proposition (2.5) depend only on the group structure, not on the Riemannian metric. Thus the geometry is in some sense completely encoded in the *coadjoint* operators: the group coadjoint $\mathrm{Ad}^*_\eta \colon \mathfrak{g}^s \to \mathfrak{g}^s$ defined so that

$$\langle \mathrm{Ad}^*_\eta v, w \rangle = \langle v, \mathrm{Ad}_\eta w \rangle \text{ for all } w \in \mathfrak{g}^s, \tag{3.1}$$

and the Lie algebra coadjoint $\mathrm{ad}^*_u \colon \mathfrak{g}^s \to \mathfrak{g}^s$ defined so that

$$\langle \mathrm{ad}^*_u v, w \rangle = \langle v, \mathrm{ad}_u w \rangle \text{ for all } w \in \mathfrak{g}^s. \tag{3.2}$$

In this section we will review how the coadjoint operators appear in the geodesic equation and the Jacobi equation, as well as computing them explicitly in the three cases of Proposition (2.5).

Remark 3.1. Note that our use of "coadjoint" for these operators is nonstandard; typically the coadjoint is defined on the *dual* of the Lie algebra $\mathfrak{g}$ by the formulas above; our version is the result on $\mathfrak{g}$ under the usual isomorphism between $\mathfrak{g}$ and its dual under the weak Riemannian metric. Our reason for doing this is to avoid the unfortunate accident of terminology in which several completely unrelated objects are all referred to as "adjoint": the alternative would be to refer to Ad_η^* as the metric adjoint of the group adjoint, which is eventually rather awkward.

In [Ar1] Arnold developed a general framework to study geodesic equations of left- as well as right-invariant metrics on arbitrary (possibly infinite dimensional) Lie groups as Euler equations on the associated Lie algebras. The next theorem is well known and easy to derive formally; we provide a short proof to keep the paper relatively self-contained.[1]

Theorem 3.2. *If G is a Lie group equipped with a (possibly weak) right-invariant metric $\langle \cdot, \cdot \rangle$ then a curve $\eta(t)$ is a geodesic if and only if the curve $u(t)$ in T_eG, given by the flow equation*

$$\dot{\eta}(t) = dR_{\eta(t)} u(t), \tag{3.3}$$

satisfies the Euler equation

$$\dot{u}(t) = -\,\mathrm{ad}_{u(t)}^* \, u(t). \tag{3.4}$$

Proof. In order to derive the equation on T_eG we write the energy of a one-parameter family of curves $(-\epsilon, \epsilon) \ni \sigma \to \eta(\sigma, t)$ with fixed endpoints at $t = a$ and $t = b$ in the form

$$E(\sigma) = \frac{1}{2} \int_a^b \|\eta_t(\sigma, t)\|^2 \, dt = \frac{1}{2} \int_a^b \big\langle dR_{\eta(\sigma,t)^{-1}} \eta_t(\sigma, t), dR_{\eta(\sigma,t)^{-1}} \eta_t(\sigma, t) \big\rangle \, dt$$

using right-invariance of the metric. Letting $y(t) = dR_{\eta(t)^{-1}}\big(\partial_\sigma \eta(\sigma, t)\big)|_{\sigma=0}$ denote the right translation to T_eG of the associated variation field along the curve η (corresponding to $\sigma = 0$) we find that

$$\partial_\sigma\big|_{\sigma=0}\big(dR_{\eta(\sigma,t)^{-1}} \partial_t \eta(\sigma, t)\big) = \partial_t y - \mathrm{ad}_u \, y.$$

Differentiating the energy functional E with respect to the parameter σ, then integrating by parts and using the fact that $y(a) = y(b) = 0$, we obtain

$$E'(0) = \int_a^b \langle u, \partial_t y - \mathrm{ad}_u \, y \rangle \, dt = -\int_a^b \langle \partial_t u + \mathrm{ad}_u^* \, u, y \rangle \, dt.$$

Since the variation field y in T_eG is arbitrary, applying Hamilton's principle yields the Euler equation in (3.4). □

Remark 3.3. If G is any one of the groups of Sobolev diffeomorphisms G^s described in the previous section then various complications arise. The most serious is the fact that the decomposition into (3.3) and (3.4) tends to lose derivatives. More precisely, since elements of T_eG^s are vector fields on M of class H^s and (for a typical right-invariant metric) the coadjoint representation involves differentiation, the result $\mathrm{ad}_u^* \, u$ will often

[1] The result can be justified for groups of smooth diffeomorphisms using Fréchet space techniques, see e.g., [O], [CKKT], [KLT].

not be a vector field of class H^s even if u is. Thus, in order to establish rigorous results of the type of Theorem 3.2 we will later impose certain smoothness conditions on the metrics as well as the topology of the underlying space.

The fact that the geodesic equation on a Lie group decouples into (3.3) and (3.4) allows us to reduce it from a second-order equation to a first-order equation, using the group coadjoint.

Corollary 3.4. *If $t \mapsto \eta(t)$ is a curve in G with velocity field $t \mapsto u(t) \in \mathfrak{g}$ satisfying equations (3.3) and (3.4) with initial conditions $\eta(0) = e$ and $u(0) = u_o$, then we have the conservation law*[2]

$$\mathrm{Ad}^*_{\eta(t)}\, u(t) = u_o. \tag{3.5}$$

As a result, we can rewrite the flow equation (3.3) as

$$\dot\eta(t) = dR_{\eta(t)}\,\mathrm{Ad}^*_{\eta(t)^{-1}}\, u_o = dL^*_{\eta(t)^{-1}} u_o. \tag{3.6}$$

Proof. The definitions (2.4) and (2.5) imply the formula

$$\frac{d}{dt}(\mathrm{Ad}_\eta) = \mathrm{ad}_{dR_{\eta^{-1}}(\dot\eta)}\,\mathrm{Ad}_\eta,$$

which immediately yields

$$\frac{d}{dt}(\mathrm{Ad}^*_\eta) = \mathrm{Ad}^*_\eta\,\mathrm{ad}^*_{dR_{\eta^{-1}}(\dot\eta)}\,.$$

With the help of this formula we can write the initial value problem for the Euler equation (3.4) on T_eG^s in the form

$$\frac{d}{dt}\big(\mathrm{Ad}^*_{\eta(t)} u(t)\big) = 0, \qquad u(0) = u_o.$$

Formula (3.5) immediately follows. See for example [AK] for discussion of the applications of this formula. □

Remark 3.5. Equation (3.6) gives the "particle equation" in Lagrangian form. In most cases, the operator $(\mathrm{Ad}_{\eta^{-1}})^*$ is a nonlocal operator (for example, for volumorphisms this equation is an integrodifferential equation, which was studied extensively in Majda-Bertozzi [MB]). This form of the equation is equivalent to the second-order geodesic equation (3.3) and (3.4). It is sometimes convenient to work with (3.6) directly to prove existence and uniqueness results; this was the basis of Kato's original proof of global existence for $\mathcal{D}_\mu(M^2)$ [K]. However this technique works better in specially constructed topologies and relies on a fairly concrete representation of the operator A^{-r}. Hence for the proof of local existence and uniqueness in Section 4, we will work with the full second-order equation. On the other hand the form (3.6) leads quite naturally to the Fredholmness result, as we shall see in Section 6.

Now let us compute the coadjoint formulas explicitly.

---

[2]For the group of volumorphisms, this is the conservation of vorticity; for the group $G = SO(3)$ describing rigid body motion, this is conservation of angular momentum.

Proposition 3.6. *Suppose G^s is the diffeomorphism group, the exact volumorphism group, or the Bott-Virasoro group. Then the Lie group coadjoint* (3.1) *and Lie algebra coadjoint* (3.2) *are given by the following formulas:*

- *If $G^s = \mathcal{D}^s(M)$ with H^r metric given by* (2.13)*, then the group coadjoint is given by*

$$\mathrm{Ad}_\eta^* v = A^{-r}\left[J(\eta)D\eta^\dagger(A^r v)\right], \tag{3.7}$$

where $J(\eta) = \det(D\eta)$ and $(D\eta)^\dagger$ is the pointwise adjoint of $D\eta$, while the Lie algebra coadjoint is given by

$$\mathrm{ad}_u^* v = A^{-r}\left[\nabla_u A^r v + (\operatorname{div} u)(A^r v) + (\nabla u)^\dagger (A^r v)\right]. \tag{3.8}$$

Here $(\nabla u)^\dagger$ is the pointwise adjoint of the operator $v \mapsto \nabla_v u$, defined so that $\langle(\nabla u)^\dagger(v), w\rangle = \langle v, \nabla_w u\rangle$ for all vectors v and w in TM.

- *If G^s is the exact volumorphism group $\mathcal{D}^s_{\mu,ex}(M)$ with right-invariant H^r metric defined by an operator A^r, then the group coadjoint is given by*

$$\begin{cases} \mathrm{Ad}_\eta^* \operatorname{sgrad} g = \operatorname{sgrad} A^{-r}\Delta^{-1}\big((A^r\Delta g)\circ\eta^{-1}\big) & \text{if } \dim(M)=2,\\ \mathrm{Ad}_\eta^* \operatorname{curl} w = \operatorname{curl} A^{-r}\Delta^{-1}\eta_*^{-1}(A^r\Delta w) & \text{if } \dim(M)=3,\\ \mathrm{Ad}_\eta^* (\delta\beta)^\sharp = \big(\delta\Delta^{-1}A^{-r}\eta^*(A^r\Delta\beta)\big)^\sharp & \text{in general.}\end{cases} \tag{3.9}$$

The Lie algebra coadjoint is given by

$$\mathrm{ad}_u^* v = \begin{cases} \operatorname{sgrad} A^{-r}\Delta^{-1}\langle u, \nabla \operatorname{curl}(A^r v)\rangle & \text{if } \dim(M)=2,\\ \operatorname{curl} A^{-r}\Delta^{-1}[u, \operatorname{curl}(A^r v)] & \text{if } \dim(M)=3,\\ (\delta A^{-r}\Delta^{-1} d\iota_u d A^r v^\flat)^\sharp & \text{in general.}\end{cases} \tag{3.10}$$

- *If G^s is the Bott-Virasoro group with right-invariant H^r metric given by* (2.15)*, then*

$$\mathrm{Ad}^*_{(\eta,\rho)}(v,q) = \big(A^{-r}\big(\eta'^2\,[A^r v]\circ\eta + qS_\eta\big), 0\big), \tag{3.11}$$

where S_η is the Schwarzian derivative (2.11)*. The Lie algebra coadjoint is*

$$\mathrm{ad}^*_{(u,p)}(v,q) = \big(A^{-r}(uA^r v' + 2u'A^r v + qu'''), 0\big). \tag{3.12}$$

Proof. To compute these, we simply use the definitions (3.1) and (3.2) along with Proposition 2.5.

- First, for the group coadjoint on $T_e\mathcal{D}^s(M)$, we have

$$\langle \mathrm{Ad}_\eta^* v, w\rangle_{H^r} = \langle v, \mathrm{Ad}_\eta w\rangle_{H^r} = \big\langle v, D\eta(w)\circ\eta^{-1}\big\rangle_{H^r} = \int_M \langle A^r v, D\eta(w)\circ\eta^{-1}\rangle\, d\mu$$
$$= \int_M \langle [A^r v]\circ\eta, D\eta(w)\rangle J(\eta)\, d\mu = \langle D\eta^\dagger([A^r v]\circ\eta), w\rangle_{L^2}.$$

Since this is true for every $w \in \mathfrak{g}^s$, formula (3.7) follows.

The computation for the Lie algebra coadjoint on $T_e\mathcal{D}^s(M)$ is similar. We have

$$\begin{aligned}\langle \mathrm{ad}_u^* v, w\rangle_{H^r} &= \langle v, \mathrm{ad}_u w\rangle_{H^r} = \langle v, -[u,w]\rangle_{H^r} = \int_M \langle A^r v, -[u,w]\rangle\, d\mu \\ &= -\int_M \langle A^r v, \nabla_u w\rangle\, d\mu + \int_M \langle A^r v, \nabla_w u\rangle\, d\mu \\ &= -\int_M \operatorname{div} u \langle A^r v, w\rangle\, d\mu + \int_M \langle \nabla_u A^r v, w\rangle\, d\mu \\ &\qquad + \int_M \langle (\nabla u)^\dagger (A^r v), w\rangle\, d\mu.\end{aligned}$$

Formula (3.8) then follows by the same reasoning as above.

- Now suppose G^s is the group of exact volumorphisms defined in Definition 2.2, so $\mathfrak{g}^s$ is the set of vector fields v expressible as $v = (\delta\beta)^\sharp$ for some 2-form β.

The general computation for the group coadjoint is straightforward from (2.8): if $w = (\delta\gamma)^\sharp$ then

$$\begin{aligned}\langle \mathrm{Ad}_\eta^* v, w\rangle_{H^r} &= \langle v, \mathrm{Ad}_\eta w\rangle_{H^r} = \int_M \langle A^r v^\flat, \delta\star(\eta^{-1})^*\star\gamma\rangle\, d\mu \\ &= \int_M \langle A^r dv^\flat, \star(\eta^{-1})^*\star\gamma\rangle\, d\mu = \int_M A^r dv^\flat \wedge \star\star(\eta^{-1})^*\star\gamma \\ &= \int_M A^r dv^\flat \wedge (\eta^{-1})^*\star\gamma = \int_M \eta^* A^r dv^\flat \wedge \star\gamma \\ &= \int_M \langle \eta^* A^r dv^\flat, \gamma\rangle\, d\mu = \langle A^{-r}\Delta^{-1}\eta^* A^r dv^\flat, \gamma\rangle_{H^r},\end{aligned}$$

so that formula (3.9) follows from (2.14).

The unusual special case when $n = 3$ comes from the following: if $w = \operatorname{curl}\xi$, then

$$\begin{aligned}\langle \mathrm{Ad}_\eta^* v, w\rangle_{H^r} &= \langle A^r v, \mathrm{Ad}_\eta \operatorname{curl}\xi\rangle_{L^2} = \int_M \langle A^r v, \operatorname{curl}((\eta^{-1})^*\xi^\flat)^\sharp\rangle\, d\mu \\ &= \int_M \langle \operatorname{curl} A^r v, ((\eta^{-1})^*\xi^\flat)^\sharp\rangle\, d\mu = \int_M (\eta^{-1})^*\xi^\flat(\operatorname{curl} A^r v)\, d\mu \\ &= \int_M \xi^\flat(\eta_*^{-1}\operatorname{curl} A^r v)\circ\eta^{-1}\, d\mu = \int_M \langle \xi, \eta_*^{-1}\operatorname{curl} A^r v\rangle\, d\mu.\end{aligned}$$

For the Lie algebra coadjoint, the easiest thing is to work directly, supposing that $w = (\delta\gamma)^\sharp$:

$$\begin{aligned}\langle \operatorname{ad}_u^* v, w\rangle_{H^r} &= -\langle A^r v, [u,w]\rangle_{L^2} \\ &= \int_M dA^r v^\flat(u,w) - u(\langle A^r v, w\rangle) + w(\langle A^r v, u\rangle)\,d\mu \\ &= \int_M \langle \iota_u dA^r v^\flat, \delta\gamma\rangle\,d\mu = \int_M \langle d\iota_u dA^r v^\flat, \gamma\rangle\,d\mu\end{aligned}$$

and the general case of formula (3.10) follows using (2.14). The special case when $\dim(M) = 2$ comes from the fact that then

$$d\iota_u dA^r v^\flat = d(\operatorname{curl}(A^r v)\star u^\flat) = \star\operatorname{div}(\operatorname{curl}(A^r v)u) = \star\langle u, \nabla\operatorname{curl}(A^r v)\rangle$$

since $\operatorname{div} u = 0$. The case when $\dim(M) = 3$ comes from the fact that $(\iota_u dA^r v^\flat)^\sharp = -u\times\operatorname{curl} A^r v$, as well as the fact that $\operatorname{curl}(u\times w) = -[u,w]$ for divergence-free vector fields u and w.

- The computations for the Bott-Virasoro group are straightforward, using the same technique as on the full diffeomorphism group, and we omit them.

□

Corollary 3.7. *For the diffeomorphism group, volumorphism group, or Bott-Virasoro group, the Euler equation* (3.4) *takes the following forms:*

- *If G^s is the full diffeomorphism group $\mathcal{D}^s(M)$ with H^r right-invariant metric, then the Euler equation is given by*

$$\begin{cases} A^r u_t + 2u_\theta A^r u + uA^r u_\theta = 0 & \text{if } \dim(M) = 1 \\ A^r u_t + \nabla_u A^r u + (\nabla u)^\dagger(A^r u) + (\operatorname{div} u)A^r u = 0 & \text{in general.}\end{cases} \tag{3.13}$$

- *If G^s is the exact volumorphism group $\mathcal{D}^s_{\mu,ex}(M)$ with H^r metric given by* (2.14)*, then the Euler equation is given by*

$$\begin{cases} \Delta A^r f_t + \{f, \Delta A^r f\} = 0 & \text{if } \dim(M) = 2, \\ \operatorname{curl} A^r u_t + [u, \operatorname{curl} A^r u] = 0 & \text{if } \dim(M) = 3, \\ A^r\Delta\beta_t + d\iota_u A^r\Delta\beta = 0 & \text{in general.}\end{cases} \tag{3.14}$$

- *If G^s is the Bott-Virasoro group with H^r metric given by* (2.15)*, then the Euler equation is given by*

$$\partial_t[A^r u] + 2u' A^r u + uA^r u' + pu''' = 0, \tag{3.15}$$

with $\partial_t p = 0$.

Proof. These formulas all follow from the Lie algebra coadjoint formulas (3.8), (3.10), and (3.12), after some minor simplifications. □

Remark 3.8. In special cases, the equations in Corollary 3.7 reduce to some well-known partial differential equations:

- If $G^s = \mathcal{D}^s(S^1)$ and $A^0 = \mathrm{id}$, then equation (3.13) reduces to Burgers' equation

$$u_t + 3uu_\theta = 0. \tag{3.16}$$

If $G^s = \mathcal{D}^s(M)$, (3.13) is called the template matching equation,

$$u_t + \nabla_u u + (\nabla u)^\dagger(u) + (\operatorname{div} u)u = 0, \tag{3.17}$$

which is used in image recognition; see [HMA]. If $G^s = \mathcal{D}^s(S^1)$ and $A^1 = \mathrm{id} - \partial_\theta^2$, then (3.13) becomes the Camassa-Holm equation ([CH] and [FF])

$$u_t - u_{t\theta\theta} + 3uu_\theta - 2u_\theta u_{\theta\theta} - uu_{\theta\theta\theta} = 0. \tag{3.18}$$

In higher dimensions, it is known as "EPDiff," or the averaged template-matching equation [HMA].

- On the exact volumorphism group $\mathcal{D}^s_{\mu,\mathrm{ex}}(M)$, if $A^0 = \mathrm{id}$ we get the usual Euler equation for ideal incompressible fluids:

$$\begin{cases} \omega_t + u(\omega) = 0 & \text{if } \dim(M) = 2, \\ \omega_t + [u, \omega] = 0 & \text{if } \dim(M) = 3, \end{cases} \tag{3.19}$$

where the vorticity is $\omega = \operatorname{curl} u$.

If $A^1 = \mathrm{id} + \alpha^2 \operatorname{curl}^2$ for some constant α, we obtain the Lagrangian-averaged Euler-α equation studied by Shkoller [Shk] and others: the equations are the same as (3.19), with the vorticity modified to $\omega = \operatorname{curl}(u + \alpha^2 \operatorname{curl}^2 u)$.

- On the Bott-Virasoro group, if $A^0 = \mathrm{id}$, equation (3.15) becomes the Korteweg-de Vries (KdV) equation

$$u_t + 3uu_\theta + pu_{\theta\theta\theta} = 0 \tag{3.20}$$

for some constant p; the fact that the KdV equation arises from a geodesic equation was first discovered by Ovsienko and Khesin [OK]. In the case where $A^1 = \mathrm{id} - \partial_\theta^2$, equation (3.15) becomes a variant of the Camassa-Holm equation

$$u_t - u_{t\theta\theta} + 3uu_\theta - uu_{\theta\theta\theta} - 2u_\theta u_{\theta\theta} + pu_{\theta\theta\theta} = 0. \tag{3.21}$$

4. Exponential map and its geodesics

One major advantage in using Lie group methods in analysis of nonlinear evolution PDE of the type we consider here lies in an important technical gain. While the Euler equation (3.4) is generally a nonlinear PDE involving unbounded operators, which requires ad hoc techniques to prove existence and uniqueness of solutions, it can be written in terms of the flow η using (3.3). The resulting (geodesic) equation is second-order in time and much more nonlinear than (3.4) (due to the presence of compositions), but in many important cases its right hand side is bounded provided that we pick the correct function space topology. The geodesic equation becomes then an ODE defined on an infinite-dimensional manifold, so that the corresponding Cauchy problem can be solved using Banach contractions and the solutions depend smoothly on the initial data. In our case this results in a differentiable exponential map.

Theorem 4.1. *Suppose $G^s = \mathcal{D}^s(M)$ is the H^s diffeomorphism group with right-invariant H^r metric given by (2.13). If r is an integer with $r \geq 1$ and $s > 2r + n/2$, then the metric and connection are both C^∞. Therefore the Riemannian exponential map is C^∞, and in particular gives a local diffeomorphism at the identity.*

Sketch of proof. The basic technique was developed by Ebin and Marsden [EM] in their study of the Cauchy problem for the Euler equations of hydrodynamics, which grew out of the original work of Gunter [Gu], Lichtenstein [Li], and Wolibner [W]. It has since been used to prove local well-posedness of many other evolution PDE[3]. Therefore we will only give an outline of the argument[4].

First we need to establish smoothness of the metric. If $A^r = \sum_{k=0}^r D_k^* D_k$ for some k^{th}-order differential operators D_k, then we can write the right-invariant metric on vectors $u, v \in T_\eta G^s$ as

$$\begin{aligned}\langle u, v\rangle_{H^r} &= \sum_{k=0}^r \int_M \langle D_k(u\circ\eta^{-1}), D_k(v\circ\eta^{-1})\rangle\, d\mu \\ &= \sum_{k=0}^r \int_M \langle dR_\eta D_k dR_{\eta^{-1}} u, dR_\eta D_k dR_{\eta^{-1}} v\rangle J(\eta)\, d\mu.\end{aligned}$$

Now if D is a first-order differential operator on vector fields, then $\overline{D}_\eta \equiv dR_\eta D dR_{\eta^{-1}}$ is continuous as a map from vector fields of class H^m to those of class H^{m-1}, as long as η is a diffeomorphism of class $s > n/2 + 1$ and $0 < m \leq s$; see Ebin [Eb] for the original proof of this, or [EM] or [Shk] for additional details. Furthermore, in the natural coordinate chart on $\mathcal{D}^s(M)$, we can write explicitly

$$\overline{D}_\eta u = (Du\circ\eta^{-1}\cdot D\eta\circ\eta^{-1})\circ\eta = Du\cdot(D\eta)^{-1},$$

where the dots represent summations over indices. Although the map $\eta \to \eta^{-1}$ is *not* smooth, the map $\eta \to D\eta$ *is* smooth, as is the operation of matrix inversion, since η is a diffeomorphism, while multiplication of functions is smooth jointly in both variables as long as it is continuous. If $s > r + n/2$, we can iterate this process to obtain smoothness for $dR_\eta D_k dR_{\eta^{-1}}$. In addition, the Jacobian $J(\eta)$ is continuous if $s > n/2 + 1$, so that the multiplication appearing in the integrand is smooth as well.

Showing that the corresponding geodesic equation is an ODE involves the same techniques. We use the formula $u = \eta_t \circ \eta^{-1}$ and $(u_t + \nabla_u u)\circ\eta = \frac{D}{dt}\frac{d\eta}{dt}$ to write the equation (3.13) in the form

$$\begin{aligned}\frac{D}{dt}\frac{d\eta}{dt} = -A^{-r}\Big(\big[\operatorname{div}\eta_t\circ\eta^{-1} + (\nabla\eta_t\circ\eta^{-1})^\dagger\big]\, A^r(\eta_t\circ\eta^{-1})\Big)\circ\eta \\ - A^{-r}\Big((\nabla_{\eta_t\circ\eta^{-1}} A^r(\eta_t\circ\eta^{-1}) - A^r(\nabla_{\eta_t\circ\eta^{-1}}(\eta_t\circ\eta^{-1}))\Big)\circ\eta.\end{aligned}\tag{4.1}$$

[3] See for example [Ma], [Shk], [Mi4], [CK] or [DKT] and references in these papers.

[4] For full details on the smoothness properties we will use refer to [Eb] and to [CK], who work out the same proof in detail for the special case $M = S^1$.

Observe that the terms in square brackets involve first-order differential operators, conjugated by R_η, and hence are smooth by the above reasoning. The operator $A^r(\eta_t \circ \eta^{-1}) \circ \eta$ is smooth as a map into vector fields of class H^{s-2r}. Furthermore the terms $\operatorname{div} u$ and $(\nabla u)^\dagger$ are in H^{s-1}, and the product of terms in H^{s-1} and H^{s-2r} is again in H^{s-2r} since $s-1 > n/2$. (This fails if $r=0$, and that is why the theorem is not true in this case: the right side of the geodesic equation is not even continuous in η.) Smoothness follows since all the terms involve multiplication and the application of $dR_\eta A^{-2r} dR_{\eta^{-1}}$ maps smoothly back into H^s by general principles of bundle maps.

The second line of (4.1) is somewhat more complicated, but uses the same techniques as above. One computes the commutator explicitly and sees that all terms involve multiplications of certain η-conjugated derivatives of η_t, which are bounded in the Sobolev topology by the assumption that $s > 2r + n/2$.

Once we have established smoothness of the right hand side of (4.1) the result follows from the fundamental theorem of ODE's on Banach manifolds, see e.g., Lang [La]. □

Remark 4.2. In the above argument, notice that smoothness of the metric is both easier and requires less of s. In particular, if $s > n/2+1$ and $r = s$, one can show with a bit more work that the right-invariant H^s Sobolev metric is smooth on TG^s. Since this metric actually generates the topology on $T_\eta G^s$ for each η, it is strong Riemannian. Hence smoothness of the connection and existence of an exponential map follows from general principles. See [La] for the details of infinite-dimensional Riemannian geometry in strong metrics.

The assumption that $s > 2r + n/2$ is thus too restrictive. In fact, we expect that the two conditions $s > n/2 + 1$ and $s \geq r$ are together sufficient. The alternative Lagrangian formulation of (3.13)

$$\frac{D}{dt}\overline{A^r_\eta}\frac{d\eta}{dt} + (\overline{\operatorname{div}_\eta}(\eta_t) + \overline{\nabla^\dagger_\eta}(\eta_t))\overline{A^r_\eta}\eta_t = 0,$$

where $\overline{D}_\eta = dR_\eta \circ D \circ dR_{\eta^{-1}}$ for an operator D, suggests that one may be able to avoid commutators as long as one can make sense of vector fields in H^{s-2r} spaces (since $s-2r$ may be negative, this may involve duals to Sobolev spaces). We will leave this for future work, however.

Theorem 4.3. *Suppose $G^s = \mathcal{D}^s_{\mu,ex}(M)$ is the group of H^s exact volumorphisms as in Definition 2.2, with right-invariant H^r metric given by* (2.14)*. If $r \geq 0$ is an integer and $s > 2r + 1 + n/2$, then the metric and connection are both C^∞. Therefore the Riemannian exponential map is C^∞, and in particular gives a local diffeomorphism at the identity.*

Sketch of proof. Since the right-invariant H^r metric on $\mathcal{D}^s_\mu(M)$ is obtained by restriction of the H^r metric on $\mathcal{D}^s(M)$, smoothness of the metric follows from Theorem 4.1.

For the existence argument, we use a different form of the geodesic equation (3.14). Generally in Riemannian geometry [La] the geodesic equation on a submanifold is the projection of the geodesic equation in the ambient manifold. Hence we can simply orthogonally project equation (3.13) onto the space of vector fields of the form $v =$

$(\delta\beta)^\sharp$. By the Hodge decomposition [EM], the H^r-projection of any vector onto the image of δ is

$$P_{H^r}(w) = P(w) = (\delta(d\delta + \delta d)^{-1} dw^\flat)^\sharp.$$

This gives the following alternate form of the geodesic equation

$$A^r u_t + P(\nabla_u A^r u + (\nabla u)^\dagger (A^r u)) = 0, \tag{4.2}$$

using the fact that $\operatorname{div} u = 0$, which in turn leads to the Lagrangian form

$$\frac{D}{dt}\frac{d\eta}{dt} = -P_\eta\Big[A^{-r}\Big(\big[\operatorname{div}\eta_t\circ\eta^{-1} + (\nabla\eta_t\circ\eta^{-1})^\dagger\big]\circ\eta\ (A^r(\eta_t\circ\eta^{-1})\circ\eta\Big) - A^{-r}\Big((\nabla_{\eta_t\circ\eta^{-1}} A^r(\eta_t\circ\eta^{-1}) - A^r(\nabla_{\eta_t\circ\eta^{-1}}(\eta_t\circ\eta^{-1}))\Big)\circ\eta\Big]. \tag{4.3}$$

Smoothness of the right hand side for $r \geq 1$ follows now again as in Theorem 4.1 and from smoothness of the projection operator $P_\eta = dR_\eta \circ P \circ dR_{\eta^{-1}}$, which was first established in [EM].

For $r = 0$, there is an additional cancelation in (4.2) coming from the formula

$$(\nabla u)^\dagger(u) = \frac{1}{2}\nabla\langle u, u\rangle,$$

which, as a gradient, vanishes after orthogonal projection onto the space of divergence-free vector fields. Hence there is no derivative loss.[5] □

Theorem 4.4. *Suppose $G^s = Vir^s(S^1)$ is the H^s Bott-Virasoro group with right-invariant H^r metric given by (2.15). If r is an integer with $r \geq 2$ and $s > 2r$, then the metric and connection are both C^∞. Therefore the Riemannian exponential map is C^∞, and in particular gives a local diffeomorphism.*

This theorem was proved by Constantin et al. [CKKT].

Remark 4.5. It should be noted that even if $r < 1$ in Theorem 4.1 or $r < 2$ in Theorem 4.4, we may still have existence and uniqueness of solutions. For example, when $r = 0$ on $\mathcal{D}^s(S^1)$, one can construct the solutions of Burgers' equation implicitly, and when $r = 1$ on $\mathrm{Vir}^s(S^1)$, one can prove directly that solutions of the KdV equation exist. However, in these cases the exponential map is continuous but not even C^1, see e.g., [CKKT]. The existence of a smooth exponential map allows us to use the techniques of finite-dimensional Riemannian geometry and study conjugate points in terms of the Jacobi equation. We turn to this in the next section.

Finally, let us record for later purposes the following obvious fact.

Corollary 4.6. *Let G^s be either $\mathcal{D}^s(M)$, $\mathcal{D}^s_\mu(M)$, or $Vir^s(S^1)$, and suppose r and s satisfy the hypotheses of Theorems 4.1, 4.3, or 4.4 respectively. For any $u_o \in T_eG^s$ the map*

$$d\exp_e(tu_o)\colon T_{tu_o}T_eG^s \simeq T_eG^s \to T_{\exp_e(tu_o)}G^s$$

[5] For more details of these computations when $r = 0$, see [EM]; when $r = 1$, see [Shk].

is a bounded linear operator satisfying $d\exp_e(0) = I$*. Consequently, the exponential map is a local diffeomorphism in a neighborhood of the identity in* G^s*.*

As in finite dimensions $\exp_e$ maps rays from the origin in T_eG^s to geodesics in G^s preserving their initial velocities. Its derivative at origin is therefore the identity map and the result follows at once from the inverse function theorem for Banach manifolds, see e.g., Lang [La].

5. Conjugate points and the Jacobi equation

We proceed to study the singularities of the exponential map, i.e., the conjugate points. The structure and distribution of these points in an abstract Hilbert manifold can be very complicated. First of all, unlike in finite-dimensional geometry, there are two possible mechanisms for the failure of the derivative of the exponential map to be an isomorphism between tangent spaces. We say that a point q is *monoconjugate* to p along a geodesic joining them if the kernel of $d\exp_p$ (considered as a linear operator from the tangent space at p to that at q) is nonempty. q is called *epiconjugate* to p if the range of this operator is a proper (not necessarily closed) subspace of the tangent space at q. Roughly speaking, the former are responsible for the minimizing properties of geodesics while the latter are related to the covering properties of the exponential map. Furthermore, again in contrast with finite dimensions, it is possible to find examples of conjugate points with conjugacies of infinite order and which have accumulation points along finite geodesic segments, as well as points which are epiconjugate but not monoconjugate.

Example 5.1. Any point on the unit sphere in the space ℓ^2 of square-summable sequences equipped with the induced metric is monoconjugate of infinite order to its antipodal point along any great circle joining them.

Example 5.2 (Grossman [Gr])**.** Consider the ellipsoid in ℓ^2 defined by

$$\mathcal{E} = \Big\{ \{x_n\} \in \ell^2 : \sum_{n=1}^{\infty} a_n x_n^2 = 1 \text{ where } \ a_1 = a_2 = 1 \ \text{ and } \ 0 < a_3 < \cdots < a_n \nearrow 1 \Big\}$$

and let $\gamma(t) = (\cos t, \sin t, 0, \dots)$ be the geodesic of the metric induced from ℓ^2. Then there is a sequence $\gamma(\pi/\sqrt{a_n})$ of monoconjugate points whose limit point $\gamma(\pi)$ is epiconjugate to $\gamma(0)$ but *not* monoconjugate.

Example 5.3. Similar phenomena arise on diffeomorphism groups. On any volumorphism group, a one-parameter subgroup of isometries γ is always a geodesic, and one can compute all conjugate points along such a geodesic explicitly. On $\mathcal{D}^s_\mu(D^2 \times S^1)$, [EMP] found a sequence of monoconjugate points along a geodesic of rigid rotation which approach an epiconjugate point. [P2] showed that on $\mathcal{D}^s_\mu(S^3)$, the point $\gamma(q\pi)$ is monoconjugate to e for every rational $q \geq 1$ of infinite order, and that for every real $t \geq \pi$, $\gamma(t)$ is epiconjugate to e. [P3] showed that such phenomena are typical for geodesics in $\mathcal{D}^s_\mu(M^3)$.

For a deeper study of conjugate points in G^s, we turn to the Jacobi equation, whose solutions give us precise information about the differential of the exponential map of the right-invariant H^r metric (2.13). Consider a geodesic $\eta(t) = \exp_e(tu_o)$ starting from the identity in the direction $u_o \in T_e G^s$. Then the Jacobi field $J(t)$ along η satisfying $J(0) = 0$ and $\dot{J}(0) = w_o$ is defined by

$$J(t) \equiv d\exp_e(tu_o)tw_o, \qquad \text{for any } w_o \in T_e G^s.$$

In the remainder of this section we will use formal arguments to derive the Jacobi equation along η directly on the tangent space at the identity as a linearization of the corresponding geodesic equation (cf. Theorem 3.2). This will produce a convenient formula for the solution operator of the Jacobi equation (Theorem 5.6 below). We will justify these constructions in Section 6 and Section 7.

Proposition 5.4. *Suppose G is any Lie group with a (possibly weak) right-invariant metric. Let $\eta(t)$ be a smooth geodesic with $\eta(0) = e$ and $\dot{\eta}(0) = u_o$. Then every Jacobi field $J(t)$ along η satisfies the following system of equations on $T_e G$:*

$$\frac{dy}{dt} - \operatorname{ad}_u y = z, \qquad \frac{dz}{dt} + \operatorname{ad}_u^* z + \operatorname{ad}_z^* u = 0, \tag{5.1}$$

where $J(t) = dR_{\eta(t)} y(t)$ and $\dot{\eta}(t) = dR_{\eta(t)} u(t)$ as in (3.3).

Proof. Let $\eta_\sigma(t) = \exp_e t(u_o + \sigma w_o)$ be a smooth variation of $\eta(t)$ through geodesics in the group. Set $y(t) = dR_{\eta(t)^{-1}}\big(\partial_\sigma|_{\sigma=0}\eta_\sigma(t)\big)$ and $z(t) = \partial_\sigma|_{\sigma=0} dR_{\eta_\sigma(t)^{-1}}\partial_t \eta_\sigma(t)$. Then differentiating equations (3.3) and (3.4) with respect to the parameter σ and evaluating at $\sigma = 0$ we obtain (5.1). □

If $G = G^s$ then in general the operators appearing in the equations (5.1) may not be bounded in the Sobolev norms, since applying the Lie algebra adjoint ad_u as well as its coadjoint ad_u^* will typically involve derivative loss as explained in Remark 3.3. On the other hand, when the exponential map is C^1 then its derivative is a bounded linear map in the H^s topology (Corollary 4.6) and the Jacobi equation will be well-posed (see also Section 8 below).[6]

We proceed next to rewrite the Jacobi equation in terms of the coadjoint representations, as we did for the geodesic equation.

Proposition 5.5. *Let G and $\eta(t)$ be as in Proposition 5.4. Then every Jacobi field $J(t)$ along η satisfies the following system on $T_e G$*

$$\frac{dw}{dt} = v, \qquad \frac{d}{dt}\big(\operatorname{Ad}_\eta^* \operatorname{Ad}_\eta v\big) + \operatorname{ad}_v^* u_o = 0 \tag{5.2}$$

where $J(t) = dL_{\eta(t)} w(t)$.

[6] Precisely, we approximate η and hence u by C^∞ geodesics to make sense of (5.1) for y and z in H^s, then use a limiting procedure and Corollary 4.6 to show that the results are still valid if η and u are only H^s.

Proof. We first rewrite equations (5.1) using the group adjoint operator as

$$\frac{d}{dt}\big(\mathrm{Ad}_{\eta(t)^{-1}} y(t)\big) = \mathrm{Ad}_{\eta(t)^{-1}} z(t), \qquad \frac{d}{dt}\big(\mathrm{Ad}^*_{\eta(t)} z(t)\big) + \mathrm{Ad}^*_{\eta(t)}\, \mathrm{ad}^*_{z(t)}\, u(t) = 0$$

and then simplify the second of these equations with the help of the conservation law (3.5) and the identity

$$\begin{aligned}\langle \mathrm{Ad}^*_{\eta(t)}\, \mathrm{ad}^*_{z(t)}\, u(t), v\rangle &= \langle u(t), \mathrm{ad}_{z(t)}\, \mathrm{Ad}_{\eta(t)}\, v\rangle = \big\langle u(t), \mathrm{Ad}_{\eta(t)}\, \mathrm{ad}_{\mathrm{Ad}_{\eta(t)^{-1}} z(t)}\, v\big\rangle \\ &= \big\langle\, \mathrm{ad}^*_{\mathrm{Ad}_{\eta(t)^{-1}} z(t)}\, \mathrm{Ad}^*_{\eta(t)}\, u(t), v\big\rangle = \big\langle\, \mathrm{ad}^*_{\mathrm{Ad}_{\eta(t)^{-1}} z(t)}\, u_o, v\big\rangle\end{aligned}$$

valid for any v in T_eG. Setting $w(t) = \mathrm{Ad}_{\eta(t)^{-1}}\, y(t)$ and $v(t) = \mathrm{Ad}_{\eta(t)^{-1}}\, z(t)$ the system (5.1) becomes (5.2). □

Therefore, the differential of the Riemannian exponential map at $tu_o \in T_eG$ applied to the vector $tw_o \in T_eG$ may now be expressed in different ways as

$$(d\exp_e)_{tu_o}(tw_o) = J(t) = dR_{\exp_e(tu_o)} y(t) = dL_{\exp_e(tu_o)} w(t) \tag{5.3}$$

where $y(t)$ solves the system (5.1) and $w(t)$ solves (5.2) with

$$y(0) = w(0) = 0, \qquad \partial_t y(0) = \partial_t w(0) = w_o, \tag{5.4}$$

and where $J(t)$ is the Jacobi field along η satisfying the same initial conditions.

Denoting the solution operator of the Cauchy problem (5.2) and (5.4) by

$$\Phi(t)\colon T_eG \to T_eG, \qquad \Phi(t)(w_o) = w(t)$$

we now proceed as in [EMP] and derive (formally) an integral equation for $\Phi(t)$.

Theorem 5.6. *Let G and $\eta(t)$ be as in Proposition 5.4. Let $\Lambda(t)$ and K_{u_o} be the linear operators on T_eG defined by the formulas*

$$\Lambda(t)(w) = \mathrm{Ad}^*_{\eta(t)}\, \mathrm{Ad}_{\eta(t)}(w) \tag{5.5}$$

and

$$K_{u_o}(w) = \mathrm{ad}^*_w\, u_o. \tag{5.6}$$

Then the solution operator $\Phi(t) = tdL_{\eta(t)^{-1}}(d\exp_e)_{tu_o}$ satisfies the equation

$$\Phi(t) = \Omega(t) + \int_0^t \Lambda(\tau)^{-1} K_{u_o} \Phi(\tau)\, d\tau \tag{5.7}$$

where

$$\Omega(t) = \int_0^t \Lambda(\tau)^{-1}\, d\tau. \tag{5.8}$$

Proof. In terms of $\Lambda(t)$ and K_{u_o} equation (5.2) becomes

$$\frac{d}{dt}\big(\Lambda(t)v(t)\big) + \frac{d}{dt}\big(K_{u_o} w(t)\big) = 0$$

which integrates to

$$\Lambda(t)v(t) = w_o - K_{u_o} w(t)$$

since $v(0) = w_o$ and $w(0) = 0$. Solving this equation for $v(t)$ and integrating once again in time we obtain

$$w(t) = \int_0^t \Lambda(\tau)^{-1} w_o \, d\tau + \int_0^t \Lambda(\tau)^{-1} K_{u_o} w(\tau) \, d\tau,$$

which implies (5.7) since $w(t) = \Phi(t)(w_o)$. □

6. Fredholmness in the weak H^r topology

Our main result in this section imposes conditions on the Sobolev indices r and s which guarantee that the differential of the exponential map of the H^r metric on the various diffeomorphism groups G^s is weakly Fredholm. In this section we do not assume that r is an integer.

Recall that a bounded linear operator between Banach spaces $T\colon X \to Y$ is *weakly Fredholm*[7] if

$$\dim \ker T < \infty \qquad \text{and} \qquad \dim Y \big/ \overline{\operatorname{ran} T}^{Y} < \infty. \tag{6.1}$$

Clearly, every Fredholm operator is weakly Fredholm. On the other hand it is not difficult to check that the bounded linear map $T\big(\{x_n\}_{n=1}^{\infty}\big) = \{x_n/n\}_{n=1}^{\infty}$ on the space of square-summable sequences ℓ^2 satisfies both conditions in (6.1) yet its range is not closed in ℓ^2. A more pertinent example is the following.

Example 6.1. Consider the ellipsoid from Example 5.2. Explicit calculations with Jacobi fields along $\gamma(t)$ show that $\gamma(\pi)$ is epiconjugate but not monoconjugate to $\gamma(0)$, while the range of $d\exp_{\gamma(0)}(\pi\dot\gamma(0))$ is a dense subspace of $T_{\gamma(\pi)}\mathcal{E}$. It follows that both numbers in (6.1) are zero and hence the differential of the exponential map at p is weakly Fredholm.

To proceed we need to justify the constructions of the previous section. In what follows $T_eG^r = \overline{T_eG^s}^{H^r}$ will denote the completion of the space T_eG^s in the H^r-norm. We begin by showing that the operators introduced formally in Theorem 5.6 are well-defined continuous maps on T_eG^r enjoying certain additional properties.

Lemma 6.2. *Let G^s be either the full diffeomorphism group $\mathcal{D}^s(M)$, the group of exact volumorphisms $\mathcal{D}^s_{\mu,ex}(M)$, or the Bott-Virasoro group $Vir^s(S^1)$, equipped with a right-invariant H^r metric given by* (2.13), (2.14)*, or* (2.15)*. Assume that $r > r_o$ (see Lemma 6.3 below), $s > n/2 + |r| + 1$ for $\mathcal{D}^s(M)$ or $\mathcal{D}^s_\mu(M)$, and $s > 5/2$ for $Vir^s(S^1)$. Then, for any $\eta \in G^s$ the maps $\mathrm{Ad}_\eta, \mathrm{Ad}^*_\eta$ and Λ are bounded invertible operators on T_eG^r.*

Proof. Recall from (2.6) that the adjoint representation of the full diffeomorphism group $G^s = \mathcal{D}^s(M)$ is given by the formula $\mathrm{Ad}_\eta(v) = \eta_* v$. To show that Ad_η is bounded it is sufficient to estimate the (right-invariant) H^r norm of the expression $D\eta \circ \eta^{-1} v \circ \eta^{-1}$ in every coordinate chart U on M, using smooth partitions of unity.

[7] We refer to Gonzalez and Harte [GH] for a detailed discussion of weakly Fredholm operators and their "index" defined as the difference of the two numbers in (6.1).

Since Zygmund functions[8] of class C_*^σ are (pointwise) multipliers of any Sobolev space $H^\rho(U)$ whenever $\sigma > \max(\rho, -\rho)$, choosing $\sigma = s - n/2 - 1$ we obtain

$$\| \operatorname{Ad}_\eta v\|_{H^r} \simeq \|D\eta\, v\|_{H^r} \lesssim \|D\eta\|_{C_*^\sigma} \|v\|_{H^r} \lesssim \|\eta\|_{H^s} \|v\|_{H^r}$$

where the last inequality is a consequence of the Sobolev embedding theorem for Hölder-Zygmund spaces, see e.g., [T1], Appendix A.

It follows that Ad_η has a unique extension to a bounded linear operator on the Hilbert space T_eG^r. Furthermore, it is invertible with $\operatorname{Ad}_\eta^{-1} = \operatorname{Ad}_{\eta^{-1}}$. Consequently its adjoint (with respect to the H^r-inner product on T_eG^r) is also invertible and satisfies $(\operatorname{Ad}_\eta^*)^{-1} = \operatorname{Ad}_{\eta^{-1}}^*$. In addition, their product $\Lambda = \operatorname{Ad}_\eta^* \operatorname{Ad}_\eta$ is self-adjoint on T_eG^r and we have equality of operator norms

$$\|\operatorname{Ad}_\eta\|_{L(H^r)} = \|\operatorname{Ad}_\eta^*\|_{L(H^r)} = \|\operatorname{Ad}_\eta^* \operatorname{Ad}_\eta\|_{L(H^r)}^{1/2} = \|\Lambda\|_{L(H^r)}^{1/2}.$$

The result follows for $\mathcal{D}^s(M)$, and the result for $\mathcal{D}_\mu^s(M)$ is an obvious consequence.

It remains to consider the case when G^s is the Bott-Virasoro group. However, recall from (2.10) the formula

$$\operatorname{Ad}_{(\eta,\rho)}(v, q) = \left(\eta_* v, q + \int_{S^1} S_\eta(\theta) v(\theta)\, d\theta \right)$$

and note that we can rewrite the integral above using (2.11) as

$$-\int_{S^1} \left(\frac{\eta'' v'}{\eta'} + \frac{\eta''^2 v}{2\eta'^2} \right) d\theta.$$

If $\eta \in H^s$ then η'' is in H^{s-2} so that η''^2 is in L^2 as long as $s - 2 > 1/2$, i.e., $s > 5/2$. In this case we also have $\eta' \in C^1$, so that $\frac{1}{\eta'}$ is in C^1 since η is a diffeomorphism. Thus the integral is bounded as long as $v \in H^1$ (a condition satisfied whenever $r > r_o$, see Lemma 6.3 below). $\square$

Lemma 6.3. *For each group G^s considered, there is a critical value r_o such that when $r > r_o$ and s is sufficiently large, then for each u in T_eG^s the operator $K_u \colon T_eG^r \to T_eG^r$ defined in* (5.6) *is compact.*

The critical values r_o and the values of s needed are given as follows:

- *When $G^s = \mathcal{D}^s(M)$, $r_o = \frac{1}{2}$ and $s > |r-1| + 2r + 1 + n/2$.*
- *When $G^s = \mathcal{D}_\mu^s(M^2)$, $r_o = -\frac{1}{2}$ and $s > 3|r| + 3$.*
- *When $G^s = \mathcal{D}_\mu^s(M^n)$ for $n \geq 3$, $r_o = 0$ and $s > 3r + 1 + n/2$.*
- *When $G^s = Vir^s(S^1)$, $r_o = \frac{3}{2}$ and $s > 3r + 1/2$.*

Proof. First, consider the case when $G^s = \mathcal{D}^s(M)$ is the full diffeomorphism group. By formula (3.8) we have

$$K_u(v) = A^{-r}\left(\nabla_v(A^r u) + (\operatorname{div} v) A^r u + (\nabla v)^\dagger (A^r u) \right).$$

[8]For details on Hölder-Zygmund spaces and their basic properties see for example Triebel [Tr] or Taylor [T1].

To prove that K_u is compact it is enough to show that in any coordinate chart U as above the map $v \mapsto \nabla_v(A^r u) + (\operatorname{div} v)A^r u + (\nabla v)^\dagger(A^r u)$ is bounded as an operator on the spaces of $\mathbb{R}^n$-valued functions from $H^r(U)$ to $H^{r-1}(U)$. It will then follow that K_u is bounded from $H^r(U)$ into $H^{3r-1}(U)$ and hence compact into $H^r(U)$ by the Rellich Lemma, since $r > 1/2 = r_0$ by assumption.

Fixing a coordinate chart and using multiplier estimates in Sobolev spaces as in the previous proof we can bound each of the three terms above separately. First, observe that $\nabla_v(A^r u)$ is tensorial in v and therefore for any $\epsilon > 0$ we have

$$\|\nabla_v(A^r u)^\flat\|_{H^{r-1}} \lesssim \|(A^r u)^\flat\|_{C^{|r-1|+1+\epsilon}}\|v\|_{H^{r-1}} \lesssim \|u\|_{C^{|r-1|+2r+1+\epsilon}}\|v\|_{H^{r-1}} \tag{6.2}$$

and, similarly, we obtain

$$\|(\operatorname{div} v + (\nabla v)^\dagger)A^r u\|_{H^{r-1}} \lesssim \|A^r u\|_{C^{|r-1|+\epsilon}}\|\operatorname{div} v\|_{H^{r-1}} \lesssim \|u\|_{C^{|r-1|+2r+\epsilon}}\|v\|_{H^r},$$

(see e.g., [Tr]). Hence all three terms are bounded by

$$\lesssim \|u\|_{C^{|r-1|+2r+1+\epsilon}}\|v\|_{H^r} \lesssim \|u\|_{H^s}\|v\|_{H^r}$$

provided that $s > n/2 + |r-1| + 2r + 1$ and ϵ is chosen suitably small.

Next, we use the same technique for the subgroup of exact volumorphisms $G^s = \mathcal{D}^s_{\mu,\mathrm{ex}}(M^2)$. In this case we have from (3.10) that

$$\operatorname{ad}_v^* u = \operatorname{sgrad} A^{-r}\Delta^{-1}\langle v, \nabla \operatorname{curl}(A^r u)\rangle$$

and therefore

$$\|\langle v, \nabla \operatorname{curl}(A^r u)\rangle\|_{H^r} \lesssim \|\nabla \operatorname{curl} A^r u\|_{C^{r+\epsilon}}\|v\|_{H^r} \lesssim \|u\|_{C^{3r+2+\epsilon}}\|v\|_{H^r},$$

so that

$$\|\operatorname{ad}_v^* u\|_{H^{3r+1}} \lesssim \|u\|_{C^{3r+2+\epsilon}}\|v\|_{H^r}.$$

Hence as long as $r > r_o = -\frac{1}{2}$ and $s > 3|r| + 2 + n/2 = 3|r| + 3$, the operator K_u maps H^r into H^{3r+1} and is hence compact by the Rellich Lemma.

In dimension three or higher, the same technique using the formula

$$\operatorname{ad}_v^* u = (\delta A^{-r}\Delta^{-1} d(\iota_v(dA^r u^\flat)))^\sharp$$

yields

$$\|\iota_v(dA^r u^\flat)\|_{H^r} \leq \|u\|_{C^{3r+1+\epsilon}}\|v\|_{H^r},$$

so that

$$\|\operatorname{ad}_v^* u\|_{H^{3r}} \lesssim \|u\|_{C^{3r+1+\epsilon}}\|v\|_{H^r},$$

and we get compactness for $s > 3r + 1 + n/2$ and $r > r_o = 0$.

For the Bott-Virasoro group, we use formula (3.12)

$$\operatorname{ad}_{(v,q)}^*(u,p) = \big(A^{-r}(vA^r u' + 2v'A^r u + pv'''), 0\big)$$

so that from above we have

$$\|A^{-r}(vA^r u' + 2v'A^r u)\|_{H^{3r-1}} \lesssim \|u\|_{C^{|r-1|+2r+1+\epsilon}}\|v\|_{H^r}$$

and thus we only need to worry about the term arising from the central extension. However, if $p \neq 0$, we clearly have

$$\|A^{-r}(pv''')\|_{H^{3r-3}} \lesssim \|v\|_{H^r},$$

and so we conclude that K_u is compact on H^r as long as $r > r_o = \frac{3}{2}$ and $s > |r-1| + 2r + \frac{3}{2} = 3r + \frac{1}{2}$. □

One can obtain stronger statements for higher Sobolev metrics, for example that K_u is Hilbert-Schmidt or trace class for sufficiently large r. We will not make any use of these observations in this paper however.

Remark 6.4. While the linear operators K_u may make sense for r small or even negative, the well-posedness results Theorems 4.1–4.4 require more, and Fredholmness will only make sense when there is a C^1 exponential map. Hence the conditions under which the well-posedness results are valid in addition to Lemma 6.2 and Lemma 6.3 are that r and s are integers, and in addition:

- for the full diffeomorphism group $G^s = \mathcal{D}^s(M)$, we need $r \geq 1$ and $s > 3r+n/2$.
- for the two-dimensional exact volumorphism group $G^s = \mathcal{D}^s_{\mu,\mathrm{ex}}(M^2)$, we need $r \geq 0$ and $s > 3r + 3$.
- for the higher-dimensional exact volumorphism group $G^s = \mathcal{D}^s_{\mu,\mathrm{ex}}(M^n)$, we need $r \geq 1$ and $s > 3r + n/2 + 1$.
- for the Bott-Virasoro group $G^s = \mathrm{Vir}^s(S^1)$, we need $r \geq 2$ and $s > 3r + 1/2$.

We are now ready to state the main result of this section.

Theorem 6.5. *Let G^s be $\mathcal{D}^s(M)$, $\mathcal{D}^s_\mu(M)$, or $Vir^s(S^1)$. Assume that r and s satisfy the conditions of Remark 6.4. For any $u_o \in T_eG$ (i.e., C^∞ smooth) let $\eta(t) = \exp_e(tu_o)$ be the geodesic of the H^r metric* (2.13) *defined on $[0,T)$ for some $T > 0$. Then the differential $d\exp_e(tu_o)\colon T_eG^s \to T_{\eta(t)}G^s$ is a weakly Fredholm operator for every $t \in [0,T)$.*

Proof. We already know that $d\exp_e(tu_o)$ is continuous in the H^s topology (see Corollary 4.6) and therefore we only need to establish the two conditions in (6.1). Furthermore, since u_o is smooth, Theorems 4.1–4.4 imply that $\eta(t)$ is also smooth and hence the differential of the left translation $dL_{\eta(t)}$ is bounded and has a bounded inverse on TG^s. Thus, showing that $d\exp_e(tu_o)$ has finite dimensional kernel in T_eG^s is equivalent to showing that the same is true for the solution operator $\Phi(t) = tdL_{\eta(t)^{-1}} \circ d\exp_e(tu_o)$ (cf. Theorem 5.6) considered as a bounded linear map on T_eG^s.

We proceed indirectly and first consider the unique bounded extension of the solution operator (which we continue to denote by $\Phi4(t)$) to the completion $T_eG^r = \overline{T_eG^s}^{H^r}$. By Lemma 6.2 the first term on the right hand side of (5.7) is a well-defined bounded operator on T_eG^r. Furthermore, for any w in T_eG^r we have

$$\begin{aligned}\langle w, \Omega(t)w\rangle_{H^r} &= \int_0^t \langle w, \Lambda(\tau)^{-1}w\rangle_{H^r}\, d\tau = \int_0^t \left\langle \operatorname{Ad}^*_{\eta(\tau)^{-1}} w, \operatorname{Ad}^*_{\eta(\tau)^{-1}} w\right\rangle_{H^r} d\tau \\ &\geq \left(\int_0^t \frac{d\tau}{\left\| \operatorname{Ad}^*_{\eta(\tau)}\right\|^2_{H^r}}\right) \|w\|^2_{H^r}\end{aligned} \tag{6.3}$$

which implies that $\Omega(t)$ is bounded below (and self-adjoint) and hence invertible on T_eG^r for each $0 < t < T$.

Regarding the second term in (5.7), observe that since K_{u_o} is compact on T_eG^r by Lemma 6.3, boundedness of the operators $\Phi(\tau)$ and $\Lambda(\tau)^{-1}$ implies compactness of the composition $\Lambda(\tau)^{-1}K_{u_o}\Phi(\tau)$ for every $0 \leq \tau \leq t$. Consequently, since the integral

$$\Gamma(t) = \int_0^t \Lambda(\tau)^{-1}K_{u_o}\Phi(\tau)\, d\tau, \qquad 0 < t < T \tag{6.4}$$

is a limit of Riemann sums, and each sum is a compact operator, $\Gamma(t)$ is also compact as a linear map on T_eG^r.

Therefore $\Phi(t) = tdL_{\eta(t)^{-1}}d\exp_e(tu_o)\colon T_eG^r \to T_eG^r$ is a sum of an invertible operator and a compact operator, and hence Fredholm. In particular, since $\ker d\exp_e(tu_o) = \ker \Phi(t)$ we conclude that the kernel of $d\exp_e(tu_o)$ is a finite-dimensional subspace of $T_eG^s \subset T_eG^r$.

Next, in order to show that $d\exp_e(tu_o)$ satisfies the second condition in (6.1) we will compute the H^s adjoint of the operator $\Psi(t) = tdR_{\eta(t)^{-1}}d\exp_e(tu_o)\colon T_eG^s \to T_eG^s$ and show that its kernel is also finite-dimensional. The proof (valid for any Hilbert manifold with a weak Riemannian metric) of the following identity can be found for example in [Gr].

Lemma 6.6. *If $\eta(t) = \exp_e(tu_o)$, then for any $w \in T_eG^s$ and $v \in T_{\eta(t)}G^s$ we have*

$$\langle d\exp_e(tu_o)w, v\rangle_{H^r} = \langle w, d\exp_{\eta(t)}(-t\dot\eta(t))v\rangle_{H^r}.$$

Since $\Psi(t)$ is bounded on T_eG^s it suffices to carry out the calculation assuming that $w_1, w_2 \in T_eG$ are C^∞ smooth. Using Lemma 6.6 we have[9]

$$\begin{aligned}\langle w_1, \Psi(t)(w_2)\rangle_{H^s} &= \langle A^{s-r}w_1, \Psi(t)(w_2)\rangle_{H^r} = \left\langle A^{s-r}w_1, tdR_{\eta(t)^{-1}}d\exp_e(tu_o)w_2\right\rangle_{H^r} \\ &= \left\langle td\exp_{\eta(t)}(-t\dot\eta(t))dR_{\eta(t)}A^{s-r}w_1, w_2\right\rangle_{H^r} = \langle \Psi(t)^*A^{s-r}w_1, w_2\rangle_{H^r} \\ &= \langle A^{-s+r}\Psi(t)^*A^{s-r}w_1, w_2\rangle_{H^s}.\end{aligned}$$

The (densely defined) operator defined by the above identity

$$A^{-s+r}\Psi(t)^*A^{s-r} = tA^{-s+r}d\exp_{\eta(t)}(-t\dot\eta(t))dR_{\eta(t)}A^{s-r}$$

[9]As in the previous sections we reserve the notation L^* for the adjoint of L computed with respect to the H^r inner product.

extends to the unique H^s adjoint of $\Psi(t)$ on T_eG^s. Using right invariance of the H^r metric (and hence its exponential map) we can rewrite this formula further as

$$tA^{-s+r}dR_{\eta(t)}d\exp_e(-u(t))A^{s-r}$$

since $\dot\eta(t) = dR_{\eta(t)}u(t)$. This formula[10] shows that the kernel of the H^s-adjoint of $\Psi(t)$ is a finite dimensional subspace of T_eG^s. Consequently, the kernel of the H^s-adjoint of $d\exp_e(tu_o)$ must also be finite dimensional. The theorem follows since

$$T_{\eta(t)}G^s = \ker d\exp_e(tu_o) \oplus_{H^s} \overline{\operatorname{ran} d\exp_e(tu_o)}^{H^s}. \tag{6.5}$$

□

It is worth stating separately another result established in the course of the above proof because of its independent interest.

Corollary 6.7. *The (unique) bounded extension of $d\exp_e(tu_o)$ to a linear map between T_eG^r and $T_{\eta(t)}G^r$ is a Fredholm operator.*

In particular, $\operatorname{ran}(d\exp_e(tu_o))$ is a closed subspace of $T_{\eta(t)}G^r$.

Remark 6.8. Corollary 6.7 was proved in [EMP] in the case when $r = 0$ and $G^s = \mathcal{D}^s_\mu(M^2)$ is the group of volume-preserving diffeomorphisms of a compact surface with or without boundary (which corresponds to 2D hydrodynamics). The argument used in the proof of Theorem 6.5 shows that in this case the associated L^2 exponential map is weakly Fredholm. However, it is still unknown whether this map is in fact Fredholm when M^2 has boundary. The major complication arises due to the derivatives normal to the boundary used in defining the H^s topology; in particular, it is not true that if u is divergence-free and tangent to the boundary, $\partial_N u$ is as well.

Remark 6.9. The relationship between weakly and strongly Fredholm maps is still not entirely clear, but some things are known. If Φ denotes $d\exp$ in the H^s topology and $\tilde\Phi$ denotes the extension to the H^r topology, one can define weakly monoconjugate points as the points in the kernel of $\tilde\Phi$ and strongly monoconjugate points as those in the kernel of Φ; similarly for weakly and strongly epiconjugate points. It is easy to see that any strongly monoconjugate point must be weakly monoconjugate, and that every strongly epiconjugate point must be weakly epiconjugate. Along smooth geodesics, we can further say that any weakly epiconjugate point is also strongly epiconjugate; see [P3]. However, other implications are unknown and possibly nontrivial.

Remark 6.10. The techniques used to prove Theorem 6.5 can easily be applied to any geodesic equation arising from a right-invariant metric on an infinite-dimensional group. In the more general setting of weak Riemannian metrics on manifolds of maps without any group structure, other techniques are required. Perhaps the simplest one is to use directly the Jacobi equation $D^2J/dt^2 + R(J,\dot\eta)\dot\eta = 0$. If the curvature operator $J \mapsto R(J,v)v$ is compact in some weaker norm for any fixed v, then Fredholmness

[10] In fact, it establishes a 1-1 correspondence between the kernel of $\Psi(t)$ and that of its H^s-adjoint. As a result, every monoconjugate point is necessarily epiconjugate.

follows as in [Mi2], using parallel translation and Egorov's theorem on conjugation by Fourier integral operators. If the curvature operator is not compact, one may still hope to separate it into positive and negative operators; only positive curvature yields conjugate points, so one might hope the positive part is a compact operator. In the following examples we show that this generally does not happen for the spaces we are considering, so that the group structure is essential for Fredholmness.

Example 6.11. Let $\mathcal{D}^s(S^1)$ be the group of H^s diffeomorphisms of the circle with $s > 3/2$ equipped with the right-invariant H^1 metric (take $r = 1$ in (2.13) above). Fix a non-zero integer k and let $u(x) = \cos kx$. Then the sectional curvature at the identity is positive and bounded below on an infinite-dimensional subspace containing the vector u. In fact, for $|l| > |k|$ we have

$$K(\cos lx, \cos kx) = \frac{\pi(1 + \frac{1}{4}k^2l^2 + \dots)}{2(1+k^2)(1+l^2)} \to \frac{\pi k^2}{8(1+k^2)}$$

as $|l| \to \infty$ (cf. [Mi4], Theorem 6.4). Therefore, there are infinitely many positive eigenvalues of the curvature operator $w \to R(w, \dot\eta)\dot\eta$ which are bounded away from zero and hence even the positive part of the operator cannot be compact. On the other hand, according to our main result in Section 7 (Theorem 7.1) the H^1 exponential map on $\mathcal{D}^s(S^1)$ is Fredholm.

Example 6.12. Let $\mathcal{D}^s_{\mu,\mathrm{ex}}(\mathbb{T}^2)$ be the group of exact volumorphisms on the 2-torus with $s > 2$ and right-invariant L^2 metric ($r = 0$ in (2.14)). By [Mi1] we can write the curvature operator as

$$R(u, v)v = P(\nabla_u \nabla p - \nabla_v \nabla q),$$

where p and q are functions such that $\Delta p = -\operatorname{div}(\nabla_v v)$ and $\Delta q = -\operatorname{div}(\nabla_u v)$.

Let V be an open set which is not all of $\mathbb{T}^2$, and suppose $v = \operatorname{sgrad} g$ with $\operatorname{supp}(g) \subset V$ and g an approximate delta function. Then it is easy to compute that $\Delta p = 2(g_{xx}g_{yy} - g_{xy}^2)$, so that $\Delta p \equiv 0$ outside V. However p will *not* be constant outside V, and so there is some open set U disjoint from V where either p_{xx} or p_{yy} is strictly positive. Suppose without loss of generality that $p_{xx} \geq \delta > 0$ on U and that $|p_{xy}| \leq M$ and $|p_{yy}| \leq M$ for some M. Letting h be a C^∞ function on $\mathbb{R}$ vanishing outside $[-1, 1]$, we can define f in a small coordinate patch inside U by the formula $f = h(x^2/\varepsilon^4 + y^2/\varepsilon^2)$, zero outside the ellipse $x^2/\varepsilon^4 + y^2/\varepsilon^2 \leq 1$. Set $u = \operatorname{sgrad} f$.

Since the supports of u and v are disjoint, we know $q \equiv 0$. Furthermore inside the ellipsoid u looks like $u = h'(x^2/\varepsilon^4 + y^2/\varepsilon^2)(\frac{y}{\varepsilon^2}\,\partial_x - \frac{x}{\varepsilon^4}\,\partial_y)$. It is then easy to see that for ε sufficiently small, we have $\langle R(u, v)v, u\rangle_{L^2} \geq \frac{\delta}{2}|u|^2_{L^2}$. Furthermore, since there are infinitely many disjoint such ellipsoids inside U, we have an infinite dimensional space on which the curvature operator is positive and bounded away from zero.

Hence Fredholmness of the exponential map for a right-invariant metric comes fundamentally from the group structure, rather than from convenient curvature properties.

7. Fredholmness in the strong H^s topology

We have already observed that the decomposition (5.7) does not work in H^s because of the loss of derivatives involved in calculating the differential of left composition with an H^s diffeomorphism and hence also in computing the group adjoint operator Ad_η etc. Therefore, to show Fredholmness of the exponential map in the (strong) H^s topology we will use a simple perturbation argument.

We prove that $d\exp_e(tu_o)\colon T_eG^s \to T_{\eta(t)}G^s$ is Fredholm whenever u_o is C^∞ smooth, using commutator estimates. Then, for any $u_o \in G^s$, we approximate $d\exp_e(tu_o)$ by a Fredholm operator with a smooth parameter and use the fact that Fredholm operators form an open subset of the space of all bounded maps from T_eG^s to $T_{\eta(t)}G^s$.

Theorem 7.1. *Let G^s be either the group of diffeomorphisms $\mathcal{D}^s(M)$, the exact volumorphism group $\mathcal{D}^s_{\mu,ex}(M)$, or the Bott-Virasoro group $Vir^s(S^1)$, with a right-invariant H^r metric. Assume that s and r are as in Remark 6.4, so that the weak Riemannian exponential map is C^1 and its differential is weakly Fredholm.*

Then for any $u_o \in T_eG^s$ the exponential map of the H^r metric (2.13) *is a nonlinear Fredholm map; i.e., its differential $d\exp_e(tu_o)\colon T_eG^s \to T_{\eta(t)}G^s$ is a Fredholm operator in the H^s topology for all t in the maximal interval of existence of $\eta(t) = \exp_e(tu_o)$.*

Proof. Suppose first that $u_o \in T_eG$, i.e., that u_o is C^∞. Then the geodesic $\eta(t)$ is in H^s for all sufficiently large s and hence is C^∞ smooth as long as it exists, say on some interval $[0,T)$. From Theorem 6.5 we already know that $d\exp_e(tu_o)$ is weakly Fredholm. Therefore our only task will be to show that $\mathrm{ran}\, d\exp_e(tu_o)$ is a closed subspace of $T_{\eta(t)}G^s$.

Consider the operator $\Omega(t)$ defined in (5.8). For any w in T_eG^s, we have

$$\begin{aligned}
\langle w, \Omega(t)w\rangle_{H^s} &= \int_0^t \big\langle w, A^{s-r}\Lambda(\tau)^{-1}w\big\rangle_{H^r}\, d\tau \\
&= \int_0^t \big\langle A^{\frac{s-r}{2}}w, \Lambda(\tau)^{-1}A^{\frac{s-r}{2}}w\big\rangle_{H^r}\, d\tau + \int_0^t \big\langle A^{\frac{s-r}{2}}w, \big[A^{\frac{s-r}{2}}, \Lambda(\tau)^{-1}\big]w\big\rangle_{H^r}\, d\tau \\
&= \int_0^t \|\mathrm{Ad}^*_{\eta(\tau)^{-1}} A^{\frac{s-r}{2}}w\|^2_{H^r}\, d\tau \\
&\qquad - \int_0^t \big\langle w, A^{-\frac{s-r}{2}}\Lambda(\tau)^{-1}\big[A^{\frac{s-r}{2}}, \Lambda(\tau)\big]\Lambda(\tau)^{-1}w\big\rangle_{H^s}\, d\tau
\end{aligned}$$

since Ad^*_η is the adjoint operator of Ad_η with respect to the H^r inner product (2.13).

For the first term on the right side above, we have

$$\|\mathrm{Ad}^*_{\eta(\tau)^{-1}} A^{\frac{s-r}{2}}w\|^2_{H^r} \geq \frac{1}{\|\mathrm{Ad}^*_{\eta(\tau)}\|^2_{L(H^r)}}\, \|A^{\frac{s-r}{2}}w\|^2_{H^r} = \frac{1}{\|\mathrm{Ad}^*_{\eta(\tau)}\|^2_{L(H^r)}}\, \|w\|^2_{H^s}. \tag{7.1}$$

Using the fact that $\Lambda(\tau) = \Lambda^*(\tau)$ is self-adjoint, we bound the second term by duality (suppressing the dependence on τ for notational simplicity)

$$\begin{aligned}\big\langle w, A^{-\frac{s-r}{2}}\Lambda^{-1}\big[A^{\frac{s-r}{2}},\Lambda\big]\Lambda^{-1}w\big\rangle_{H^s} &= \big\langle A^{\frac{s}{2}}w, A^{\frac{r}{2}}\Lambda^{-1}\big[A^{\frac{s-r}{2}},\Lambda\big]\Lambda^{-1}w\big\rangle_{L^2} \\ &\lesssim \|w\|_{H^s}\big\|\Lambda^{-1}\big[A^{\frac{s-r}{2}},\Lambda\big]\Lambda^{-1}w\big\|_{H^r} \\ &\le \|w\|_{H^s}\|\mathrm{Ad}_{\eta^{-1}}\|^2_{L(H^r)}\big\|\big[A^{\frac{s-r}{2}},\Lambda\big]\Lambda^{-1}w\big\|_{H^r}\end{aligned} \tag{7.2}$$

and then exploit the presence of the commutator as follows. First, a straightforward computation gives an explicit formula (again suppressing the dependence on τ)

$$\Lambda = P_G A^{-r} M_{J(\eta)D\eta^\dagger}\overline{A^r_\eta}M_{D\eta}$$

where as before

$$\overline{A^r_\eta}(w) = \big(A^r(w\circ\eta^{-1})\big)\circ\eta,$$

and M_C is the multiplication operator by an $n\times n$ matrix C, whose transpose is denoted by $C^\dagger$. (We use P_G to denote the L^2 projection onto T_eG^s, if G^s is the exact volumorphism group.)

Since A^r is an elliptic pseudodifferential operator in $OPS^{2r}_{1,0}$ and η is a smooth diffeomorphism of M, the operator $\overline{A^r_\eta}$ is also in $OPS^{2r}_{1,0}$. Moreover, for the volumorphism subgroup the corresponding projection P_G belongs to $OPS^0_{1,0}$. It now follows from general composition properties for such operators that $\Lambda(\tau)$ is a pseudodifferential operator of class $OPS^0_{1,0}$ on T_eG^s whose principal symbol, in any coordinate chart on M, is given by a matrix of symbols of order zero. See Taylor [T1] for proofs of these and following statements.

Consequently, since $A^{(s-r)/2}\in OPS^{s-r}_{1,0}$ has scalar principal symbol $\sigma_{A^{(s-r)/2}}(x,\xi) = (1+|\xi|^{s-r})I$, the commutator $[A^{(s-r)/2},\Lambda]$ is of class $OPS^{s-r-1}_{1,0}$. Using boundedness in Sobolev norms of the operators in $OPS^m_{1,0}$ (for any $m\in\mathbb{R}$), we can therefore further estimate the right hand side of (7.2) by

$$\|w\|_{H^s}\|\mathrm{Ad}_{\eta(\tau)^{-1}}\|^2_{L(H^r)}\|\Lambda(\tau)^{-1}w\|_{H^{s-1}} \le c(\tau)\|w\|_{H^{s-1}}\|w\|_{H^s},$$

where the constant $c(\tau)$ depends on $\eta(\tau)$. Combining this with estimates in (7.1) and (7.2) for any t in $[0,T)$ we obtain

$$\|w\|_{H^s} \le c_1\|\Omega(t)w\|_{H^s} + c_2\|w\|_{H^{s-1}} \tag{7.3}$$

where $c_1 = \int_0^t\|\mathrm{Ad}^*_{\eta(\tau)}\|^{-2}_{L(H^r)}d\tau$ and $c_2 = \int_0^t c(\tau)d\tau$.

To finish the argument in the case when u_o is C^∞ it suffices now to recall the formula for the solution operator in (5.7)

$$\Phi(t)w = \Omega(t)w + \int_0^t \Lambda(\tau)^{-1}K_{u_o}\Phi(\tau)w\,d\tau$$

which together with (7.3) yields (up to a constant depending on η) the estimate

$$\|w\|_{H^s} \lesssim \|\Phi(t)w\|_{H^s} + \|w\|_{H^{s-1}} + \|\Gamma(t)w\|_{H^s} \tag{7.4}$$

where $\Gamma(t) = \int_0^t \Lambda(\tau)^{-1} K_{u_o} \Phi(\tau)\, d\tau$ is a compact operator on $T_e G^s$ (see (6.4) in the proof of Theorem 6.5). This estimate implies that $\Phi(t)$ has closed range in $T_e G^s$. It follows that $d\exp_e(tu_o)$ has closed range in $T_{\eta(t)}G^s$ and thus in light of Theorem 6.5 is Fredholm for any $t \in [0,T)$.

Assume now that u_o (and hence η) is H^s. Approximate u_o by a smooth vector field $\tilde{u}_o$ so that $\|u_o - \tilde{u}_o\|_{H^s} < \varepsilon$, where ε will be determined momentarily. Since the exponential map is C^∞ in the H^s topology, its derivative $u_o \mapsto d\exp_e(tu_o)$ depends smoothly on u_o (from Theorems 4.1–4.4). In particular, it is locally Lipschitz and therefore satisfies $\|d\exp_e(tu_o) - d\exp_e(t\tilde{u}_o)\|_{L(H^s)} \lesssim \|u_o - \tilde{u}_o\|_{H^s}$ uniformly on any time interval $[0,T']$ with $T' < T$.

Shrinking the time interval a bit, if necessary, from the estimate in (7.4) we now immediately obtain for any w in $T_e G^s$

$$\begin{aligned}
\|w\|_{H^s} &\lesssim \|\tilde{\Phi}(t)w\|_{H^s} + \|w\|_{H^{s-1}} + \|\tilde{\Gamma}(t)w\|_{H^s} \\
&\lesssim \|d\exp_e(tu_o)w\|_{H^s} + \|d\exp_e(t\tilde{u}_o)w - d\exp_e(tu_o)w\|_{H^s} + \|w\|_{H^{s-1}} + \|\tilde{\Gamma}(t)w\|_{H^s} \\
&\lesssim \|d\exp_e(tu_o)w\|_{H^s} + \varepsilon\|w\|_{H^s} + \|w\|_{H^{s-1}} + \|\tilde{\Gamma}(t)w\|_{H^s}
\end{aligned}$$

where $\tilde{\Phi}(t) = tdL_{\exp_e(-t\tilde{u}_o)} d\exp_e(t\tilde{u}_o)$ and $\tilde{\Gamma}(t) = \int_0^t \tilde{\Lambda}(\tau)^{-1} K_{\tilde{u}_o} \tilde{\Phi}(\tau)\, d\tau$ are the operators corresponding to $\tilde{u}_o \in T_e G$. Choosing $\varepsilon > 0$ small enough we deduce from this inequality that $d\exp_e(tu_o)$ has closed range in H^s and therefore is a Fredholm operator from $T_e G^s$ into $T_{\eta(t)} G^s$.

The result follows from invariance of the index under compact perturbations and the fact that $\operatorname{ind}\Omega(t) = 0$ for any t in $[0,T)$. □

Remark 7.2. Above we have been assuming that s is substantially larger than r. If $r = s$, so that we have a strong metric on G^s, then a similar proof works to prove that if u_o and hence η are C^∞, then the differential of the exponential map along η is Fredholm in the H^s topology (the analog of the weak Fredholmness from the last section). A similar approximation procedure gives strong Fredholmness for this situation.

In the next two sections we discuss some applications of Theorem 7.

8. The Morse Index Theorem in hydrodynamics

In [AK] (Remark 6.5, p. 225) the authors asked whether the index of a finite geodesic segment in the volumorphism group with the right-invariant L^2 metric (corresponding to an ideal fluid flow) is finite. The main theorem of this section provides an affirmative answer to this question for any two-dimensional fluid flow. The theorem itself is more general and can be viewed as a hydrodynamical analogue of the Morse index theorem in finite-dimensional geometry.

The same question can be asked about geodesics of any of the diffeomorphism groups G^s equipped with the weak H^r metric (2.13). Our proof of Theorem 8.2, which follows an abstract approach of [Uh] and relies on the Fredholm result of Theorem 7.1, would differ in the general case only in minor technical details.

Using different techniques, a result of this type has been recently obtained by Biliotti, Exel, Piccione and Tausk [BEPT] for strong Riemannian Hilbert manifolds.

Our objective is to show that the index of the Hessian of the L^2 energy functional

$$E(\eta) = \frac{1}{2}\int_0^1 \|\dot{\eta}(t)\|^2_{L^2}\,dt$$

at a critical point η in $\mathcal{D}^s_\mu(M^2)$ (a 2D fluid flow) joining the identity $\eta(0) = e$ with $\eta(1)$ can be computed by counting the number of conjugate points (with their multiplicities) along the geodesic.

For any $0 \le t \le 1$ we let $\tau_t\colon T_e\mathcal{D}^s_\mu(M^2) \to T_{\eta(t)}\mathcal{D}^s_\mu(M^2)$ denote the L^2 parallel transport operator. τ_t is an isomorphism of the tangent spaces along η preserving the L^2-metric (see e.g., [Mi1]), and we use it to identify the space of smooth vector fields on η that are zero at e and $\eta(t)$ with the space of $T_e\mathcal{D}^s_\mu$-valued functions that vanish at the endpoints. We complete the latter in the norm defined by the inner product

$$\langle v, w\rangle_{\dot{H}^1_t L^2_x} = \int_0^t \langle \dot{v}(t'), \dot{w}(t')\rangle_{L^2}\,dt' \tag{8.1}$$

and denote[11] it by $H_t = H^1_0\big([0,t], T_e\mathcal{D}^0_\mu\big)$. We will consider any function in H_t to be also an element of H_{t_1} by extending it by zero to the interval $t \le t' \le t_1$. The Hessian of E can now be identified with the bounded symmetric bilinear form on $H_t \times H_t$

$$I_t(v, w) = \int_0^t \Big(\langle \dot{v}(t'), \dot{w}(t')\rangle_{L^2} - \langle R_\eta(v(t')), w(t')\rangle_{L^2}\Big)dt' \tag{8.2}$$

where $R_\eta(v) = \tau_{t'}^{-1} R\big(\tau_{t'}v, \dot{\eta}\big)\dot{\eta}$ is the L^2 curvature tensor of $\mathcal{D}^s_\mu$ along η. The symmetry of I_t is immediate from the symmetry properties of R. Moreover, we have

Lemma 8.1. *For any $v, w \in T_e\mathcal{D}^s_\mu(M^n)$ we have the following estimate*

$$\|R(w, v)v\|_{H^\sigma} \le C\|v\|^2_{H^s}\|w\|_{H^\sigma} \qquad (0 \le \sigma \le s)$$

where the constant C depends on $s > n/2 + 1$.

One immediate corollary of this lemma (and the right-invariance of the curvature tensor R) is the continuity of the bilinear form

$$|I_t(v, w)| \lesssim \big(1 + \|\dot{\eta}\circ\eta^{-1}\|^2_{L^\infty_t H^s_x}\big)\|v\|_{\dot{H}^1_t L^2_x}\|w\|_{\dot{H}^1_t L^2_x}$$

(by Cauchy-Schwarz and the fact that $\tau(t)$ is an L^2 isometry), and hence existence of an associated bounded operator[12] $v \mapsto L_t(v) = v + \partial_t^{-2} R_t(v)$ on H_t computed directly from (8.2) using (8.1) and integration by parts. Another direct consequence of Lemma 8.1 is the global well-posedness of the Jacobi equation along η.

[11] As before we use the notation $T_e\mathcal{D}^0_\mu = \overline{T_e\mathcal{D}^s_\mu}^{L^2}$.

[12] For any $v \in H_t$ we set $\partial_t^{-2}v(t') = \int_0^{t'}\int_0^r f(r')\,dr'dr - \frac{t'}{t}\int_0^t\int_0^r f(r')\,dr'dr$.

Proof. For the estimate in the case when $\sigma = s$ we refer to [Mi1]. Let $\sigma = 0$. Since $\mathcal{D}^s_\mu$ is a smooth submanifold of the full diffeomorphism group $\mathcal{D}^s$ we can express its sectional curvature using the Gauss-Codazzi equations as

$$\langle R(w,v)v,u\rangle_{L^2} = \langle \bar{R}(w,v)v,u\rangle_{L^2} + \langle Q_{L^2}\nabla_w u, Q_{L^2}\nabla_v v\rangle_{L^2} - \langle Q_{L^2}\nabla_v w, Q_{L^2}\nabla_u v\rangle_{L^2}$$

where ∇ is the covariant derivative on M, $\bar{R}$ is the L^2 curvature tensor of $\mathcal{D}^s$ and $Q_{L^2} = I - P_{L^2}$ is the Hodge projection onto the gradients.[13] Using Cauchy-Schwarz and the Sobolev lemma, we have

$$\langle \bar{R}(w,v)v,u\rangle_{L^2} = \int_M \langle R(w(x),v(x))v(x),u(x)\rangle\, d\mu \leq C\|v\|^2_{H^s}\|w\|_{L^2}\|u\|_{L^2}.$$

With the help of the formulas

$$Q_{L^2} = \nabla\Delta^{-1}\mathrm{div} \qquad \text{and} \qquad \mathrm{div}\nabla_v w = \mathrm{tr}\,(Dv\cdot Dw) + \mathrm{Ric}(v,w)$$

where $\mathrm{Ric}(v,w)$ is the Ricci curvature of M, we similarly get

$$\|Q_{L^2}\nabla_v w\|_{L^2} = \|Q_{L^2}\nabla_w v\|_{L^2} \leq C\|v\|_{H^s}\|w\|_{L^2},$$

and, for any positive α, a bound in Hölder norms

$$\|Q_{L^2}\nabla_v v\|_{C^{1+\alpha}} \leq C_\alpha\|v\|_{Lip}\|v\|_{C^{1+\alpha}}.$$

Choosing $\alpha < s - n/2 - 1$ and integrating by parts, we obtain

$$\langle Q_{L^2}\nabla_w u, Q_{L^2}\nabla_v v\rangle_{L^2} = -\langle u, \nabla_w Q_{L^2}\nabla_v v\rangle_{L^2} \leq C_\alpha\|v\|^2_{H^s}\|w\|_{L^2}\|u\|_{L^2},$$

which gives the result for $\sigma = 0$. The full estimate follows by interpolating between the L^2 and the H^s estimates. □

Recall that the index $i(t)$ of the form I_t is the dimension of the largest subspace of H_t on which it is negative definite. Let $j(t)$ be the dimension of the largest subspace on which I_t is non-positive and set $n(t) = j(t) - i(t)$. Our main result here is

Theorem 8.2. *Let $\eta(t)$, $0 \leq t \leq 1$, be a geodesic from the identity e to $\eta(1)$ in the group of volumorphisms $\mathcal{D}^s_\mu(M^2)$ of a surface without boundary. The index of the form I_t is finite and equal to the number of conjugate points to e along η each counted with multiplicity according to the formula ind$(I_1) = i(1) = \sum_{0<t<1} n(t)$.*

Theorem 8.2 will be obtained directly from the following abstract result.

Theorem 8.3 (Uhlenbeck [Uh])**.** *Let $H_t, 0 \leq t \leq 1$, be an increasing family of Hilbert spaces satisfying $\overline{\bigcup_{t'<t} H_{t'}} = H_t = \overline{\bigcap_{t'>t} H_{t'}}$. Let B be a bilinear form such that*

(1) *B is Fredholm of finite index,*
(2) *B satisfies the unique continuation property.*

Then the function $i(t)$ (resp. $j(t)$) is l.s.c. (resp. u.s.c.), $n(t) \neq 0$ at only finitely many points, and ind(B_1) − ind$(B_0) = \sum_{0<t<1} n(t)$, where B_t is the restriction of B to H_t.

[13] Note that so far we have been using the projection onto the *exact* volumorphism group; the projection onto the *full* volumorphism differs by an operator whose range is the finite-dimensional space of harmonic fields. This does not affect any of the results.

As usual, a bilinear form on a Hilbert space is called Fredholm if the associated linear operator is Fredholm. It is said to have the unique continuation property if the kernels of these associated operators are pairwise disjoint.

Proof of Theorem 8.2. The required properties of the family $H_t = H^1_0\big([0,t], T_e\mathcal{D}^0_\mu\big)$ are immediate from the definitions. Thus we only need to show that the index form I_t satisfies (1) and (2) above. Both conditions will follow from Fredholmness of the L^2 exponential map and general linear ODE principles.

Since $(L_t)^* = L_t$ is self-adjoint, in order to establish (1) it will be sufficient to show that $\dim\ker L_t < \infty$ and $\operatorname{ran} L_t \subset H_t$ is closed. However, since $v + \partial_t^{-2} R(v) = 0$ if and only if $\ddot{v} + R(v) = 0$, we see that $\ker L_t$ coincides with the kernel (of the bounded extension to $T_e\mathcal{D}^0_\mu$) of $d\exp_e(t\dot\eta(0))$ and so is finite dimensional by Corollary 6.7.

To show that the range of L_t is a closed subspace of H_t observe that, given any $w \in H_t$, finding $v \in H_t$ which satisfies $w = L_t(v)$ is equivalent to solving the non-homogeneous ODE system in $T_e\mathcal{D}^0_\mu \times T_e\mathcal{D}^0_\mu$

$$\frac{d}{dt}\begin{pmatrix} u \\ \dot u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -R & 0 \end{pmatrix}\begin{pmatrix} u \\ \dot u \end{pmatrix} - \begin{pmatrix} 0 \\ R(w) \end{pmatrix} \qquad \text{where} \quad u = v - w \tag{8.3}$$

subject to the conditions $u(0) = u(t) = 0$. Note that the right hand side of (8.3) is bounded by Lemma 8.1. Using standard ODE theory in Banach spaces we first solve the boundary value problem and then rewrite its (unique) solution $u = u_w$ (depending continuously on the parameter w) in terms of the resolvent U_t of the corresponding homogeneous Cauchy problem on $T_e\mathcal{D}^0_\mu \times T_e\mathcal{D}^0_\mu$ with initial data $u_w(0) = 0$ and $\dot u_w(0)$. We therefore get a bounded linear map $w \mapsto v$ from H_t to itself

$$v(t') = w(t') + \pi_1 U_{t'}\left(\begin{pmatrix} 0 \\ \dot u_w(0) \end{pmatrix} - \int_0^{t'} U_{-s'}\begin{pmatrix} 0 \\ R(w) \end{pmatrix} ds'\right), \qquad 0 \le t' \le t \le 1$$

where π_1 denotes the projection onto the 1st factor in $T_e\mathcal{D}^0_\mu \times T_e\mathcal{D}^0_\mu$, which implies that the range of L_t must be a closed subset of H_t for each $0 \le t \le 1$.

It remains to establish the unique continuation property for the index form I_t in condition (2). This however follows directly from the definition of the family $H_t = H^t_0\big([0,t], T_e\mathcal{D}^0_\mu\big)$ and the properties of Jacobi fields. For suppose that $t' < t$ and $v \in \ker L_t \cap \ker L_{t'}$. Then v is zero on the interval $[t', t]$ and therefore has zero initial data at t', which in turn implies by uniqueness of the solutions to the Jacobi equation that it must be identically zero on the whole interval. □

Recall that, as in finite dimensional geometry, a monoconjugate point along some geodesic γ can be thought of as the point where another geodesic, starting from the same initial point, meets γ infinitesimally. As an application of the Morse Index Theorem 8.2 we find that this is actually true locally and thus obtain a nice geometric interpretation of conjugate points in hydrodynamics.

Remark 8.4. In finite dimensional Riemannian geometry a classical theorem of Morse and Littauer states that the exponential map is never injective in any neighborhood of a conjugate point. An infinite dimensional analogue of this result was proved in [BEPT] as a corollary of their Morse Index Theorem for strong Riemannian Hilbert manifolds. We expect that the same result will hold for the L^2 exponential map on $\mathcal{D}^s_\mu(M^2)$ as a direct application of Theorem 8.2 using the same argument as in the strong Riemannian case.[14]

9. Covering properties of the exponential map

Our second application of the Fredholmness result of Section 7 is to surjectivity properties of H^r exponential maps on diffeomorphism groups. We will focus on two cases of particular interest: that of the strong Riemannian metric ($r = s$) on G^s and that of 2D hydrodynamics ($r = 0$).

If $r = s$ then G^s is a complete Riemannian Hilbert manifold and it follows from a result of Ekeland [Ek] that the set of diffeomorphisms which can be connected to the identity e by a minimal H^s geodesic in G^s contains a dense G_δ, i.e., a countable intersection of open sets. (There are many explicit examples of points in complete Riemannian Hilbert manifolds without minimizing geodesics, or even any geodesics, joining them. See e.g., [At] and [Gr].) This result can be improved using Theorem 7.1.

In the case of the L^2 metric on $\mathcal{D}^s_\mu(M^2)$ our results are somewhat less satisfying. In a pioneering paper [Shn1] Shnirelman showed that there exist configurations of a 3D ideal fluid that cannot be connected by a minimizing L^2 geodesic (see also [Shn2] or the exposition in [AK]). However, he also conjectured that this problem has a solution in 2D, see [Shn3]. Our result in Theorem 9.2 is an attempt to shed some light on this conjecture.

We will rely heavily on the constructions of Rabier [R1] adapted to our situation. Given $\exp_e\colon T_eG^s \to G^s$ we will consider the set of its generalized critical values

$$K = \big\{\eta \in G^s : \exists v_n \in T_eG^s \ \text{ such that } \lim_n \exp_e(v_n) = \eta \ \text{ and } \lim_n \nu\big(d\exp_e(v_n)\big) = 0\big\}$$

where $\nu(T)$ denotes the surjectivity modulus of a bounded linear operator T between Banach spaces. We also introduce the sets

$$K_0 = \big\{\eta \in G^s : \eta = \exp_e(v) \text{ for some } v \in T_eG^s \text{ and } \nu\big(d\exp_e(v)\big) = 0\big\}$$

and

$$K_\infty = \big\{\eta \in G^s : \exists v_n \in T_eG^s \text{ with no converging subsequence in } T_eG^s \text{ and such that } \lim_n \exp_e(v_n) = \eta \ \text{ and } \lim_n \nu\big(d\exp_e(v_n)\big) = 0\big\}$$

consisting of critical values and asymptotic critical values of $\exp_e$, respectively, as well as the set

$$\widetilde{K} = \big\{\eta \in G^s : \exists v_n \in T_eG^s \text{ such that } \exp_e(v_n) = \eta \text{ and } \lim_n \nu\big(d\exp_e(v_n)\big) = 0\big\}.$$

It is not difficult to see that K is a closed subset of G^s with $K = K_0 \cup K_\infty$ and that $K_0 \subset \widetilde{K} \subset K \cap \operatorname{ran}(\exp_e)$. Furthermore, the fact that $\exp_e$ is a Fredholm map by

[14] We thank Paolo Piccione for pointing out this result to us.

Theorem 7.1 combined with the Smale-Sard theorem [Sma] implies that K_0 must be of first Baire category in G^s.

Consider now the case when G^s is a strong Riemannian Hilbert manifold.

Theorem 9.1. *Any diffeomorphism in $U = G^s \backslash K$ can be connected to the identity e by a minimizing H^s geodesic. Furthermore, the set $K_\infty \backslash \mathrm{ran}(\exp_e)$ is of first Baire category in G^s and consists only of points whose H^s distance to e is infinite. In particular, the set of diffeomorphisms that can be joined to e contains an open and dense subset of G^s.*

Proof. First, observe that U is open and by the result of Ekeland (Theorem B, [Ek]) it is the only open connected component of the identity in G^s.

Next, since the exponential map is Fredholm we can apply Theorem 6.1 of [R1] to deduce that it defines over U a (locally trivial) C^0 fibre bundle. Thus, in particular, $\exp_e$ maps onto U. Furthermore, local trivialization ensures that the fibre $\exp_e^{-1}(\eta)$ over any $\eta \in U$ is homeomorphic to the fibre over the identity $\exp_e^{-1}(e) = \{0\} \cup S$, a disjoint union with $S \subset T_eG^s$. Since $\exp_e$ is Fredholm of index zero, there is a natural bijection between the kernel and the cokernel of its derivative. Moreover, by construction, the surjectivity modulus $\nu(d\exp_e)$ is strictly positive on $\exp_e^{-1}(U)$, and thus restricted to this set $\exp_e$ is a local diffeomorphism, by the inverse function theorem. This in turn implies that $S \cap \exp_e^{-1}(U)$ must be a discrete set without any limit points in $\exp_e^{-1}(U)$. Therefore, given any $\eta \in U$ we can always pick $v \in \exp_e^{-1}(\eta)$ with the smallest norm, and setting $\gamma(t) = \exp_e\big(tv/\|v\|_{H^s}\big)$, obtain a minimal geodesic connecting e to η.

In order to prove the second statement we will first show that $K_\infty \setminus \widetilde{K}$ is of first category in G^s. Consider the set $\Sigma = \{v \in T_eG^s : \nu(d\exp_e(v)) = 0\}$ of all critical points of $\exp_e$. We claim that this set has empty interior in T_eG^s. In fact, no ray through the origin in T_eG^s intersects Σ along a nonempty interval. Otherwise we would obtain a whole segment of conjugate points on the corresponding geodesic in G^s, contradicting the fact that such points must be isolated along finite geodesic segments in G^s by the Morse Index Theorem of Section 8. The first part of the second statement follows now from Lemma 3.2 of [R2] since $\widetilde{K} \subset \mathrm{ran}(\exp_e)$.

Finally, given $\eta \in K_\infty$ let v_n be a sequence in T_eG^s without converging subsequences as in the definition of K_∞ and let $\eta_n = \exp_e(v_n)$. Observe that we can assume that $\|v_n\|_{H^s}$ grows without bound. Otherwise, v_n would be confined to some H^s ball B in T_eG^s of sufficiently large radius, and since Fredholm maps are locally proper (see Smale [Sma]), $\exp_e \overline{B}^{H^s}$ would be a closed subset of G^s; thus $\eta_n \to \eta$ in G^s would necessarily imply that $\eta \in \exp_e \overline{B}^{H^s}$. Next, assume that the H^s distance

$$d_{H^s}(e,\eta) = \inf\big\{L^s(\omega) : \omega \text{ is a piecewise smooth curve in } G^s \text{ from } e \text{ to } \eta\big\}$$

is finite (here L^s is the length functional of the H^s metric). Given any $\epsilon > 0$ pick a curve ω_ϵ whose length satisfies $L^s(\omega_\epsilon) < d_{H^s}(e,\eta) + \epsilon$. Since $\eta_n \to \eta$ in G^s we can pick an $N > 0$ sufficiently large such that $d_{H^s}(\eta_n,\eta) < \min(\epsilon,\epsilon_1)$ whenever $n \geq N$, where $\epsilon_1 > 0$ is chosen so that $\exp_\eta \colon B(0,\epsilon_1) \subset T_\eta G^s \to G^s$ is a diffeomorphism onto some neighborhood of η. Then we have

$$d_{H^s}(\eta, \eta_n) = \| \exp_\eta^{-1}(\eta_n)\|_{H^s} < \epsilon_1$$

and hence

$$L^s(\omega_\epsilon) + d_{H^s}(\eta, \eta_n) < d_{H^s}(e, \eta) + 2\max(\epsilon, \epsilon_1).$$

On the other hand, since $d_{H^s}(e, \eta_n) = \|v_n\|_{H^s} \nearrow \infty$ we can always find n sufficiently large such that

$$d_{H^s}(e, \eta) + 2\max(\epsilon, \epsilon_1) < \|v_n\|_{H^s}.$$

However, this implies that the path

$$t \to \omega_\epsilon(t) \cup \exp_\eta\left(t \exp_\eta^{-1}(\eta_n)/\| \exp_\eta^{-1}(\eta_n)\|_{H^s}\right)$$

in G^s from e to η_n is shorter than γ_n which contradicts minimality of γ_n. □

For the volumorphism group $\mathcal{D}^s_\mu(M^2)$ equipped with the L^2 metric we get a weaker result. As before, we find that the L^2 exponential map induces a fibre bundle over the connected component of the identity and that the set $K_\infty \setminus \mathrm{ran}(\exp_e)$ is again of first Baire category. However, in this case we cannot apply Ekeland's theorem to rule out the possibility that the corresponding set K of generalized critical values separates $\mathcal{D}^s_\mu(M^2)$ into disjoint open components. Nevertheless, we will show that the open connected component U of e is in general a large subset of $\mathcal{D}^s_\mu(M^2)$. For simplicity we assume $M^2 = \mathbb{T}^2$.

Theorem 9.2. *The open connected component U of the identity e in $\mathcal{D}^s_\mu(M^2)$ has infinite L^2 diameter. Moreover, the L^2 exponential map $\exp_e\colon \exp_e^{-1}(U) \to U$ is a covering space map.*

Proof. We only need to prove the first statement. To do this, we use the following result of Eliashberg and Ratiu [ER].

If η is a symplectomorphism on M such that for some ball Q and a strictly smaller ball Q_0 we have $\eta(Q) \subset Q$ and $\eta|_{Q\setminus Q_0} \equiv \mathrm{id}$, then the L^2 distance between the identity and η is bounded below by a constant times the Calabi invariant of the restriction to Q:

$$C|\mathrm{Cal}(\eta|_Q)| \le d_{L^2}(e, \eta).$$

Fix such a ball Q in M^2 and choose polar coordinates such that $r = 0$ is the center of Q, with $r = b$ the outer radius of Q and $r = a$ the outer radius of Q_0. We can assume that $\omega = dx \wedge dy = r\,dr \wedge d\theta$. Let $u = \phi(r)\,\partial_\theta$ for some C^∞ function ϕ which is identically zero for $r \ge a$. Then u is a steady solution of the Euler equation (3.14), and so the flow $\eta(t)$ of u is a geodesic in $\mathcal{D}_\mu(M^2)$; clearly $\eta(t)$ is the identity on $M\setminus Q_0$, and $\eta(t)(Q) \subset Q$.

We compute the Calabi invariant for a symplectomorphism preserving a ball Q and fixing a neighborhood of its boundary as follows. Since Q is simply connected, the symplectic form ω is closed, with $\omega = d\lambda$ for some 1-form λ. Thus if $\eta(t)$ is a curve in the group of symplectomorphisms, we have $\eta^* d\lambda = d\lambda$, and hence $d(\eta(t)^*\lambda - \lambda) = 0$ for every t. Since $(\eta(t)^*\lambda - \lambda)$ is a closed 1-form which is exact when $t = 0$, it must be exact for all t, and hence $\eta(t)^*\lambda - \lambda = dh(t)$ for some time-dependent function $h(t)$, which vanishes on the boundary of Q. The Calabi invariant of η restricted to Q is then

$$\mathrm{Cal}(\eta(1)|_Q) = \int_Q h(1)\, d\mu.$$

In this case we have $\lambda = \frac{1}{2}(x\,dy - y\,dx) = \frac{1}{2}r^2\,d\theta$. With $u = \phi(r)\,\partial_\theta$, we then have $\eta(t)^*\lambda = \frac{tr^2}{2}\phi'(r)\,dr + \frac{r^2}{2}\,d\theta$, so that $h_r(t,r) = t\frac{r^2}{2}\phi'(r)$. Since ϕ is not constant, we clearly obtain that $|\int_Q h| = At$ for some positive constant A, and hence the distance from e to $\eta(t)$ is at least some constant multiple of t.

On the other hand, it is known that for a purely rotational flow on a flat space, the curvature operator is nonpositive [P1]. Hence there are no conjugate points along the geodesic η, and so the entire image of $\eta(t)$ is contained in U. In particular, the diameter of U is infinite. □

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