

The exponential map of the group of area-preserving diffeomorphisms of a surface with boundary

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Abstract We prove that the Riemannian exponential map of the right-invariant L^2 metric on the group of volume-preserving diffeomorphisms of a two-dimensional manifold with a nonempty boundary is a nonlinear Fredholm map of index zero.

1 Introduction

Consider a compact n -dimensional manifold M with a smooth boundary ∂M equipped with a Riemannian metric. Let \mathcal{D}_μ^s be the volumorphism group; that is, the group of diffeomorphisms of M which preserve the Riemannian volume form μ and are of Sobolev class H^s . It is well-known that if $s > n/2 + 1$, then \mathcal{D}_μ^s is a submanifold of the infinite dimensional Hilbert manifold \mathcal{D}^s of all H^s diffeomorphisms of M . Its tangent space $T_\eta \mathcal{D}_\mu^s$ consists of H^s sections X of the pull-back bundle $\eta^* TM$ whose right-translations $X \circ \eta^{-1}$ to the identity element are the divergence-free vector fields on M that are parallel to the boundary ∂M . The L^2 inner product for vector fields

$$\langle u, v \rangle_{L^2} = \int_M \langle u(x), v(x) \rangle d\mu(x) \quad u, v \in T_e \mathcal{D}_\mu^s \quad (1.1)$$

defines a right-invariant metric on \mathcal{D}^s and hence also on \mathcal{D}_μ^s with associated Levi-Civita connections. The curvature tensor \mathcal{R} of this metric on \mathcal{D}_μ^s is a bounded trilinear operator on each tangent space and is invariant with respect to right translations by \mathcal{D}_μ^s . Our main references for the basic facts about \mathcal{D}_μ^s and its L^2 geometry are the papers [7], [12], [14] and the monograph [2].

Arnold, in his pioneering paper [1], reinterpreted the hydrodynamics of an ideal fluid filling M in terms of the Riemannian geometry of the volumorphism group of M equipped with the L^2 metric describing the fluid's kinetic energy. He showed that a curve $\eta(t)$ is a geodesic of the L^2 metric on \mathcal{D}_μ^s starting from the identity element e in the direction v_0 if and only if the time dependent vector field $v = \dot{\eta} \circ \eta^{-1}$ on

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M solves the incompressible Euler equations

$$\begin{aligned}\partial_t v + \nabla_v v &= -\text{grad } p \\ \text{div } v &= 0 \\ \langle v, \nu \rangle &= 0 \text{ on } \partial M\end{aligned}\tag{1.2}$$

with the initial condition

$$v(0) = v_0\tag{1.3}$$

where p is the pressure function, ∇ denotes the covariant derivative on M and ν is the outward pointing normal to the boundary ∂M .

It turns out that there is a technical advantage in rewriting the Euler equations this way; Ebin and Marsden [7] showed that the Cauchy problem for the corresponding geodesic equation in \mathcal{D}_μ^s can be solved uniquely on short time intervals by a standard Banach-Picard iteration argument. In particular, its solutions depend smoothly on the data, and as a result one can define (at least for small t) a smooth exponential map

$$\exp_e : T_e \mathcal{D}_\mu^s \rightarrow \mathcal{D}_\mu^s, \quad \exp_e tv_0 = \eta(t),$$

where $\eta(t)$ is the unique geodesic of (1.1) issuing from the identity with initial velocity $v_0 \in T_e \mathcal{D}_\mu^s$. The exponential map is a local diffeomorphism from an open set around zero in $T_e \mathcal{D}_\mu^s$ onto a neighborhood of the identity in \mathcal{D}_μ^s . This follows from the inverse function theorem and the fact that the derivative of \exp_e at time $t = 0$ is the identity map. Furthermore, if $n = 2$ then by the classical result of Wolibner [21] the exponential map can be extended to the whole tangent space $T_e \mathcal{D}_\mu^s$, which is interpreted as geodesic completeness of the volumorphism group with respect to the L^2 metric.

The structure and distribution of singularities of the exponential map of (1.1) has been of considerable interest ever since the problem of conjugate points in \mathcal{D}_μ^s was raised by Arnold in [1]. The first examples of conjugate points were constructed in [12] and [13] in the case when M is a sphere with the round metric or the flat 2-torus. Further examples can be found in [18], [15], [16], [3] and [4]. In [8] it was proved that the L^2 exponential map is a non-linear Fredholm map of index zero whenever M is a compact manifold of dimension 2 without boundary and moreover that the Fredholm property fails for a steady rotation of the solid torus in \mathbb{R}^3 . More pathological counterexamples were constructed in [15] using curl eigenfields on the sphere \mathbb{S}^3 and more recently in [17] in the case of certain axisymmetric flows in \mathbb{R}^3 . Furthermore, Shnirelman [19] proved that when M is the flat 2-torus the exponential map on \mathcal{D}_μ^s is a Fredholm quasiruled map. In [14] the authors showed that the failure of the Fredholm property in the case of three-dimensional manifolds is ‘‘borderline,’’ in the sense that the exponential maps of Sobolev H^r metrics are necessarily Fredholm whenever $r > 0$.

An outstanding problem left unresolved in these papers concerns the case when a two-dimensional manifold M has a nonempty boundary ∂M . The methods employed in [8] allowed only for a much weaker result, namely, that the derivative of the exponential map along a geodesic in \mathcal{D}_μ^s can be extended to a linear Fredholm operator defined on the L^2 completions of the tangent spaces to the volumorphism group. The question of whether the behavior is genuinely different in case of a boundary has been raised in light of recent work where phenomena have been discovered that seem to rely heavily on the presence of the boundary (such as double-exponential growth of the vorticity field in 2D [10] and numerically-observed blowup in 3D [11]).

The main goal of the present paper is to establish the strong H^s Fredholmness property of the exponential map for incompressible 2D fluids in the presence of boundaries. For notational simplicity and clarity of exposition we will consider the simplest case of flow on a cylinder $M = S^1 \times [0, L]$ for some $L > 0$, so that we can work in a single chart. The general case of bounded domains in \mathbb{R}^2 can be treated in the standard way by choosing a suitable open cover of the boundary ∂M together with a subordinate smooth partition of unity and applying the result for the cylinder with large L . Our main result in this paper is thus the following

Theorem 1.1 *Let $M = S^1 \times [0, L]$ be the cylinder of height $L > 0$ with boundary $\partial M = S^1 \times \{0\} \cup S^1 \times \{L\}$, endowed with the Euclidean metric. For $s > 2$, the exponential map of the L^2 metric (1.1) on $\mathcal{D}_\mu^s(M)$ is a nonlinear Fredholm map of index zero.*

A direct consequence of Theorem 1.1 is that monoconjugate and epiconjugate points coincide, have finite multiplicity and cannot accumulate along finite geodesic segments. Furthermore, the exponential map on $T\mathcal{D}_\mu^s$ restricts to the same map on $T\mathcal{D}_\mu$ and geodesics remain as smooth as their initial velocity

(see [7] Theorem 12.1) so the exponential map continues to be Fredholm of index zero on \mathcal{D}_μ . In the next section we recall the basic setup from [8] and [14]. The proof of Theorem 1.1 will be given in Sections 3 and 4. The key element of the proof involves deriving lower bounds for the invertible part of the derivative dexp_e with respect to a suitably chosen Sobolev-type norm defined on the space of stream functions on the manifold. The main idea here is that we have an operator that is invertible because it is positive-definite in low Sobolev norms, but in the standard higher Sobolev norms it is not due to boundary terms; however by weighting the coefficients differently we can make the operator positive-definite in the new inner product up to leading order.

2 The setup: Jacobi fields and the exponential map

We first collect a few well-known facts about Fredholm mappings. A bounded linear operator L between Banach spaces is said to be Fredholm if it has finite dimensional kernel and cokernel. It then follows from the open mapping theorem that $\text{ran } L$ is closed. L is said to be semi-Fredholm if it has closed range and either its kernel or cokernel is of finite dimension. The index of L is defined as $\text{ind } L = \dim \ker L - \dim \text{coker } L$. L is Fredholm of index zero if and only if it can be written in the form $L = \Omega - \Gamma$ where Ω is invertible and Γ is compact. The set of semi-Fredholm operators is an open subset in the space of all bounded linear operators and the index is a continuous function on this set into $\mathbb{Z} \cup \{\pm\infty\}$, cf. Kato [9]. A C^1 map f between Banach manifolds is called Fredholm if its Fréchet derivative $df(p)$ is a Fredholm operator at each point p in the domain of f . If the domain is connected then the index of the derivative is by definition the index of f , cf. Smale [20].

Let γ be a geodesic in a Riemannian Hilbert manifold. A point $q = \gamma(t)$ is said to be conjugate to $p = \gamma(0)$ if the derivative $\text{dexp}_p(t\dot{\gamma}(0))$ is not an isomorphism considered as a linear operator between the tangent spaces at p and q . It is called monoconjugate if $\text{dexp}_p(t\dot{\gamma}(0))$ fails to be injective and epiconjugate if $\text{dexp}_p(t\dot{\gamma}(0))$ fails to be surjective. In general exponential maps of infinite dimensional Riemannian manifolds are not Fredholm. For example, the antipodal points on the unit sphere in a Hilbert space with the induced metric are conjugate along any great circle and the differential of the corresponding exponential map has infinite dimensional kernel. An ellipsoid constructed by Grossman [6] provides another example as it contains a sequence of monoconjugate points along a geodesic arc converging to a limit point at which the derivative of the exponential map is injective but not surjective. Such pathological phenomena are ruled out by the Fredholm property because in this case monoconjugate and epiconjugate points must coincide, have finite multiplicities and cannot cluster along finite geodesic segments.

Let M be a Riemannian manifold of dimension $n = 2$ with boundary ∂M and assume $s > 2$. Given any vector v_0 in $T_e \mathcal{D}_\mu^s$ let $\eta(t) = \exp_e(tv_0)$ be the geodesic of the L^2 metric starting from the identity with velocity v_0 . The derivative of the exponential map at tv_0 can be expressed in terms of the Jacobi fields. Since the curvature tensor \mathcal{R} of the L^2 metric is bounded in the H^s topology it follows that the solutions of the Jacobi equation

$$\ddot{J} + \mathcal{R}(J, \dot{\eta})\dot{\eta} = 0 \quad (2.1)$$

along $\eta(t)$ with initial conditions

$$J(0) = 0, \quad \dot{J}(0) = w_0 \quad (2.2)$$

are unique and persist (as long as the geodesic is defined) by the standard ODE theory on Banach manifolds, cf. [12]. Define the Jacobi field solution operator Φ_t by

$$w_0 \rightarrow \Phi_t w_0 = \text{dexp}_e(tv_0)tw_0 = J(t). \quad (2.3)$$

Next, recall that for any $\eta \in \mathcal{D}_\mu^s$ the group adjoint operator on $T_e \mathcal{D}_\mu^s$ is given by $\text{Ad}_\eta = dR_{\eta^{-1}}dL_\eta$ where R_η and L_η denote the right and left translations by η . Consequently, given any $v, w \in T_e \mathcal{D}_\mu^s$ we have

$$w \rightarrow \text{Ad}_\eta w = \eta_* w = D\eta \circ \eta^{-1}(w \circ \eta^{-1}) \quad (2.4)$$

and the corresponding algebra adjoint operator

$$\text{ad}_v w = -[v, w]. \quad (2.5)$$

The associated coadjoint operators are defined using the L^2 inner product by

$$\langle \text{Ad}_\eta^* v, w \rangle_{L^2} = \langle v, \text{Ad}_\eta w \rangle_{L^2} \quad (2.6)$$

and

$$\langle \text{ad}_v^* u, w \rangle_{L^2} = \langle u, \text{ad}_v w \rangle_{L^2} \quad (2.7)$$

for any u, v and $w \in T_e \mathcal{D}_\mu^s$. Our general strategy of the proof of Theorem 1 will be similar to that in the case when M has no boundary. The proof of the following result can be found in [14].

Proposition 2.1 *Let $v_0 \in T_e \mathcal{D}_\mu^s$ and let $\eta(t)$ be the geodesic of the L^2 metric (1.1) in \mathcal{D}_μ^s starting from the identity e with velocity v_0 . Then Φ_t defined in (2.3) is a family of bounded linear operators from $T_e \mathcal{D}_\mu^s$ to $T_{\eta(t)} \mathcal{D}_\mu^s$. Furthermore, if $v_0 \in T_e \mathcal{D}_\mu^{s+1}$ then Φ_t can be represented as*

$$\Phi_t = D\eta(t)(\Omega_t - \Gamma_t) \quad (2.8)$$

where Ω_t and Γ_t are bounded operators on $T_e \mathcal{D}_\mu^s$ given by

$$\Omega_t = \int_0^t \text{Ad}_{\eta(\tau)^{-1}} \text{Ad}_{\eta(\tau)^{-1}}^* d\tau \quad (2.9)$$

$$\Gamma_t = \int_0^t \text{Ad}_{\eta(\tau)^{-1}} K_{v(\tau)} dR_{\eta^{-1}(\tau)} \Phi_\tau d\tau \quad (2.10)$$

and K_v is a compact operator on $T_e \mathcal{D}_\mu^s$ given by

$$w \rightarrow K_{v(t)} w = \text{ad}_w^* v(t), \quad w \in T_e \mathcal{D}_\mu^s \quad (2.11)$$

and where $v(t)$ is the solution of the Cauchy problem (1.2)-(1.3).

Proof See [8], Prop. 4 and Prop. 8.

Remark 2.2 Note that the decomposition (2.8)-(2.11) must be applied with care. This is due to the loss of derivatives involved in calculating the differential of the left translation operator $\xi \rightarrow L_\eta \xi$ and consequently of the adjoint operator in (2.4). This is why we consider v_0 , and hence $\eta(t)$, in H^{s+1} rather than H^s .

As mentioned in the Introduction we also have the following

Proposition 2.3 *For any $v_0 \in T_e \mathcal{D}_\mu^s$ the derivative $d\exp_e(tv_0)$ extends to a Fredholm operator on the L^2 -completions $\overline{T_e \mathcal{D}_\mu^{L^2}}$ and $\overline{T_{\exp_e(tv_0)} \mathcal{D}_\mu^{L^2}}$.*

Proof A detailed proof may be found in [8], Thm. 2, but the main idea is as follows. The operator (2.9) is invertible on $\overline{T_e \mathcal{D}_\mu^{L^2}}$. This follows from Lemma 3.1 below and self-adjointness in the L^2 inner product. Compactness of the operator (2.10) on $\overline{T_e \mathcal{D}_\mu^{L^2}}$ follows from compactness of the operator K_v , and hence compactness of the composition appearing under the integral in (2.10), and finally from viewing the integral as a limit of sums of compact operators. This represents $d\exp_e(tv_0)$ as the sum of an invertible operator and compact operator which implies $d\exp_e(tv_0)$ is Fredholm of index zero.

In particular, it follows that monoconjugate points along $\eta(t)$ in \mathcal{D}_μ^s have finite multiplicity.

3 Proof of Theorem 1: Preliminary Estimates

To show that the L^2 exponential map on \mathcal{D}_μ^s is a Fredholm map we will prove that for each $t > 0$ its derivative Φ_t is a bounded Fredholm operator from $T_e \mathcal{D}_\mu^s$ to $T_{\eta(t)} \mathcal{D}_\mu^s$; that is, Φ_t can be expressed as the sum of an invertible operator and a compact operator on $T_e \mathcal{D}_\mu^s$. We will assume that the initial divergence-free vector field v_0 in (2.3) is of class C^∞ . The general H^s case will then follow from a density argument, just as in [8]. Compactness of (2.10) then follows as described in the proof of Proposition 2.3.

It remains to prove that the operator Ω_t , defined by (2.9), is invertible on the tangent space $T_e \mathcal{D}_\mu^s$. We begin with an L^2 estimate which is straightforward.

Lemma 3.1 *Assume $s > 2$. Given $v_0 \in T_e \mathcal{D}_\mu^s$ let $\eta(t) = \exp_e tv_0$ be the corresponding L^2 geodesic. For any $w \in T_e \mathcal{D}_\mu^s$ and any $t \geq 0$ we have*

$$\langle w, \Omega_t w \rangle_{L^2} \geq C_t \|w\|_{L^2}^2 \quad (3.1)$$

where Ω_t is defined by (2.9) and $C_t = \int_0^t \|D\eta(\tau)\|_\infty^{-2} d\tau$.

Proof From (2.9) we compute

$$\begin{aligned} \langle w, \Omega_t w \rangle_{L^2} &= \int_0^t \langle w, \text{Ad}_{\eta(\tau)^{-1}} \text{Ad}_{\eta(\tau)}^* w \rangle_{L^2} d\tau \\ &= \int_0^t \|\text{Ad}_{\eta(\tau)^{-1}} w\|_{L^2}^2 d\tau \geq \|w\|_{L^2}^2 \int_0^t \|\text{Ad}_{\eta(\tau)}^*\|_{L(L^2)}^{-2} d\tau \end{aligned}$$

and since Ad_η^* is an L^2 adjoint of Ad_η , formula (2.4) implies

$$\|\text{Ad}_{\eta(t)}^*\|_{L(L^2)}^2 = \|\text{Ad}_{\eta(t)}\|_{L(L^2)}^2 \leq \|D\eta(t)^T D\eta(t)\|_\infty$$

which gives (3.1).

Next, we proceed to derive the estimate in H^s norms. It will be convenient to work with stream functions on $M = S^1 \times [0, L]$. As is well-known, we may write any divergence-free $v \in T_e \mathcal{D}_\mu^s(M)$ as

$$v_f = -\partial_y f \frac{\partial}{\partial x} + \partial_x f \frac{\partial}{\partial y}, \quad (3.2)$$

for a uniquely-defined function $f \in H^{s+1}(M)$ satisfying $f(x, L) = 0$ and $f(x, 0) = c$ for some $c \in \mathbb{R}$, for all $x \in S^1$. Thus we introduce the space

$$\mathcal{F}^{s+1}(M) = \{f \in H^{s+1}(M) : \exists c \in \mathbb{R} \text{ s.t. } f(x, L) = 0 \text{ and } f(x, 0) = c \forall x \in S^1\}. \quad (3.3)$$

From (2.4) and (2.6) we have

$$v_{\Lambda_t f} = \text{Ad}_{\eta(t)}^* \text{Ad}_{\eta(t)} v_f \quad (3.4)$$

for a bounded invertible operator $\Lambda_t: \mathcal{F}^{s+1}(M) \rightarrow \mathcal{F}^{s+1}(M)$ which we will compute in Lemma 3.2. Rewriting Ω_t on the space of stream functions as $\widehat{\Omega}_t$, so that $\Omega_t v_f = v_{\widehat{\Omega}_t f}$, our goal therefore reduces to establishing the following

$$\text{Claim: For any } t > 0 \text{ the operator } f \rightarrow \widehat{\Omega}_t f = \int_0^t \Lambda_\tau^{-1} f d\tau \text{ is invertible on } \mathcal{F}^{s+1}(M), \quad (3.5)$$

with Λ_t^{-1} the inverse of Λ_t in (3.4). To this end we will proceed indirectly since the formula for Λ_t is somewhat simpler to work with than the formula for the inverse Λ_t^{-1} .

Our approach to proving the claim (3.5) is as follows. For some positive constants B_0, \dots, B_s we define a semi-inner product on \mathcal{F}^{s+1} by

$$\langle\langle f, g \rangle\rangle_{s+1} = \sum_{j=0}^s B_j \langle \partial_x^j \partial_y^{s-j} \nabla f, \partial_x^j \partial_y^{s-j} \nabla g \rangle_{L^2} \quad (3.6)$$

with associated semi-norm $\|f\|_{s+1} = \langle\langle f, f \rangle\rangle_{s+1}^{1/2}$, which is equivalent to the standard H^{s+1} seminorm on $\mathcal{F}^{s+1}(M)$ and thus to the H^s norm on $T_e \mathcal{D}^s(M)$. Then, we show that the constants B_0, \dots, B_s can be chosen so that

$$\langle\langle f, g \rangle\rangle_{s+1} \geq K \|f\|_{s+1}^2 - C \|f\|_{s+1} \|f\|_{\dot{H}^s} \quad (3.7)$$

for $g = \Lambda_t f$, where $\|f\|_{\dot{H}^s}$ denotes the homogeneous Sobolev norm defined by (3.10) below. Applying this estimate to $f = \Lambda_t^{-1} g$ allows us to derive the estimate

$$\|\widehat{\Omega}_t g\|_{s+1} \geq K_1 \|g\|_{s+1} - C_1 \|g\|_{\dot{H}^s}$$

for some positive constants C_1 and K_1 , which shows that $\widehat{\Omega}_t$ has closed range on \mathcal{F}^{s+1} . This, together with Lemma 1, implies that $\widehat{\Omega}_t$ is semi-Fredholm with trivial kernel whose index at $t = 0$ is zero. Since the index is constant on connected components of the space of semi-Fredholm operators (cf. [9]), we conclude that the index is always zero, so that $\widehat{\Omega}_t$ has trivial cokernel and is therefore invertible.¹

To carry out our plan we need to estimate the boundary terms and this is our main goal here; the analysis of these terms begins in Proposition 3 below. In the next Lemma we derive an explicit formula for the operator Λ_t defined by formula (3.4). For simplicity we will suppress the dependence on t and just write η and Λ until the time dependence matters again in Proposition 4.4.

¹ We note that if M has no boundary then the estimate (3.7) already holds with $B_0 = \dots = B_s = 1$, as shown in [8], but one can demonstrate with simple counterexamples that no such universal estimate can hold for all η if M has a boundary; the details are not terribly interesting and we will omit them here.

Lemma 3.2 *Let $f \in \mathcal{F}^{s+1}(M)$ be an H^{s+1} stream function on M as in formula (3.3), and let $\eta \in \mathcal{D}_\mu(M)$ be a smooth area-preserving diffeomorphism. Then the operator Λ defined by formula (3.4) is given, for $g = \Lambda f$, as the unique solution of the system*

$$\Delta g = \operatorname{div}(G_\eta \nabla f), \quad g(x, L) = 0, \quad g(x, 0) \text{ is constant}, \quad \int_0^{2\pi} \partial_y g(x, 0) dx = \int_0^{2\pi} |\partial_x \eta(x, 0)|^2 f(x, 0) dx,$$

$$\text{where } G_\eta = (D\eta^T D\eta)^{-1} = \begin{pmatrix} |\partial_y \eta|^2 & -\langle \partial_x \eta, \partial_y \eta \rangle \\ -\langle \partial_x \eta, \partial_y \eta \rangle & |\partial_x \eta|^2 \end{pmatrix}. \quad (3.8)$$

Proof First we establish that there exists a unique solution g of the problem (3.8); that is for any H^{s-1} function ψ on M and any constant k , there is a unique $g \in \mathcal{F}^{s+1}(M)$ satisfying $\Delta g = \psi$, $g(x, L) = 0$, $g(x, 0)$ constant, and $\int_0^{2\pi} \partial_y g(x, 0) dx = k$.

The easiest way to do this is to reduce it to a Dirichlet problem and apply well-known results. To this end we define γ to satisfy

$$\Delta \gamma = \phi, \quad \gamma(x, L) = \gamma(x, 0) = 0.$$

Since ϕ is in H^{s-1} , we know a unique solution γ exists and is in $H^{s+1}(M)$. Furthermore if we set $\zeta = g - \gamma$, then ζ satisfies the problem

$$\Delta \zeta = 0, \quad \zeta(x, L) = 0, \quad \zeta(x, 0) \text{ is constant}.$$

There is clearly a one-parameter family of solutions uniquely determined by the value of this constant m , given by $\zeta(x, y) = m(1 - y/L)$. It follows that there is a unique g given by

$$g(x, y) = \gamma(x, y) + m(1 - y/L),$$

where m is determined by the condition

$$k = \int_0^{2\pi} \partial_y \gamma(x, 0) dx - \frac{2\pi m}{L}.$$

Having shown that a unique solution of (3.8) exists, our strategy is now to show that if g solves (3.8), then $\langle v_h, v_g \rangle_{L^2} = \langle v_h, v_{\Lambda f} \rangle_{L^2}$ for every $h \in \mathcal{F}^{s+1}(M)$, which will imply that $g = \Lambda f$. For any such h , we have

$$\langle v_h, v_{\Lambda f} \rangle_{L^2} = \int_M \langle v_h, \operatorname{Ad}_\eta^* \operatorname{Ad}_\eta v_f \rangle dx dy = \int_M \langle \operatorname{Ad}_\eta v_h, \operatorname{Ad}_\eta v_f \rangle dx dy.$$

Using formula (2.4) for the adjoint action Ad_η , we have

$$\langle v_h, v_{\Lambda f} \rangle_{L^2} = \int_M \langle D\eta(v_h) \circ \eta^{-1}, D\eta(v_f) \circ \eta^{-1} \rangle dx dy = \int_M \langle v_h, (D\eta)^T(D\eta)(v_f) \rangle dx dy,$$

using the change of variables formula and the fact that η is volume-preserving.

Since $v_f = J\nabla f$ where J is the antisymmetric operator of rotation by 90° , we have

$$\langle v_h, v_{\Lambda f} \rangle_{L^2} = -\langle \nabla h, J(D\eta)^T(D\eta)J\nabla f \rangle_{L^2} = \langle G_\eta \nabla h, \nabla f \rangle_{L^2},$$

as an easy computation shows. Integrating by parts using the divergence theorem, we get

$$\langle v_h, v_{\Lambda f} \rangle_{L^2} = - \int_M h \operatorname{div}(G_\eta \nabla f) dx dy - h(\cdot, 0) \int_0^{2\pi} G_\eta(\partial_y, \partial_y)(x, 0) f_y(x, 0) dx,$$

since $h(x, 0)$ is constant.

The same integration by parts shows that

$$\langle v_h, v_g \rangle_{L^2} = - \int_M h \Delta g dx dy - h(\cdot, 0) \int_0^{2\pi} \partial_y g(x, 0) dx,$$

and this is true for every $h \in \mathcal{F}^{s+1}(M)$ if and only if g solves the system (3.8).

The following inequality appears in [8] but without the boundary terms.

Proposition 3.3 *Let $\eta \in \mathcal{D}_\mu(M)$ be a C^∞ -smooth area-preserving diffeomorphism and let $g = Af$ where $f \in \mathcal{F}^{s+1}(M)$, as in Lemma 3.2. For any nonnegative integers m and n , not both zero, let $f_{m,n} = \partial_x^m \partial_y^n f$ and $g_{m,n} = \partial_x^m \partial_y^n g$. Then we have²*

$$\begin{aligned} \langle \nabla f_{m,n}, \nabla g_{m,n} \rangle_{L^2} &\geq K_\eta \|\nabla f_{m,n}\|_{L^2}^2 - C \|\eta\|_{C^{m+n+1}}^2 \|\nabla f_{m,n}\|_{L^2} \|f\|_{\dot{H}^{m+n}} \\ &\quad + \int_{\partial M} f_{m+1,n} \partial_x^m \partial_y^{n-1} (\partial_x g - |\partial_y \eta|^2 \partial_x f + \langle \partial_x \eta, \partial_y \eta \rangle \partial_y f) dx \end{aligned} \quad (3.9)$$

where $K_\eta = \|D\eta\|_\infty^{-2}$, with C some constant independent of η , and

$$\|f\|_{\dot{H}^{s+1}} = \sum_{0 \leq i+j \leq s} \|\nabla \partial_x^i \partial_y^j f\|_{L^2}. \quad (3.10)$$

Proof We start with an integration by parts to obtain

$$\begin{aligned} \langle \nabla f_{m,n}, \nabla g_{m,n} \rangle_{L^2} &= \int_M \langle \nabla f_{m,n}, \nabla g_{m,n} \rangle dx dy \\ &= \int_M \operatorname{div} (f_{m,n} \nabla g_{m,n}) dx dy - \int_M f_{m,n} \partial_x^m \partial_y^n \Delta g dx dy. \end{aligned} \quad (3.11)$$

Using Lemma 3.2 we further rewrite the above as

$$\begin{aligned} &= \int_{\partial M} f_{m,n} \langle \nabla g_{m,n}, \nu \rangle dx - \int_M f_{m,n} \operatorname{div} (\partial_x^m \partial_y^n G_\eta \nabla f) dx dy \\ &= \int_{\partial M} f_{m,n} |_{\partial M} (\langle \nabla g_{m,n}, \nu \rangle - \partial_x^m \partial_y^n \langle G_\eta \nabla f, \nu \rangle) |_{\partial M} dx + \int_M \langle \nabla f_{m,n}, \partial_x^m \partial_y^n G_\eta \nabla f \rangle dx dy, \end{aligned} \quad (3.12)$$

where we again used the divergence theorem, recalling that the boundary is $\partial M = S^1 \times \{0\} \cup S^1 \times \{L\}$, with outward unit normals $\nu = (0, 1)$ for $y = L$ and $\nu = (0, -1)$ for $y = 0$.

We proceed to analyze these terms separately. Observe that $G_\eta = (D\eta^T D\eta)^{-1}$ is a positive-definite matrix and the last term can be written as

$$\begin{aligned} \langle \nabla f_{m,n}, \partial_x^m \partial_y^n G_\eta \nabla f \rangle_{L^2} &= \int_M \langle \nabla f_{m,n}, G_\eta \partial_x^m \partial_y^n \nabla f \rangle dx dy + \int_M \langle \nabla f_{m,n}, [\partial_x^m \partial_y^n, G_\eta] \nabla f \rangle dx dy \\ &\geq \int_M |(D\eta^T)^{-1} \nabla f_{m,n}|^2 dx dy - \|\nabla f_{m,n}\|_{L^2} \|[\partial_x^m \partial_y^n, G_\eta] \nabla f\|_{L^2}. \end{aligned}$$

Since G_η is a matrix of smooth functions, the commutator with any differential operator of order $m+n$ is a differential operator of lower order with coefficients involving derivatives of η up to order $m+n+1$ at most. Hence we have an estimate

$$\|[\partial_x^m \partial_y^n, G_\eta] \nabla f\|_{L^2} \leq C \|\eta\|_{C^{m+n+1}}^2 \|f\|_{\dot{H}^{m+n}(M)}$$

with $\|\cdot\|_{\dot{H}^{m+n}}$ denoting the Sobolev H^{m+n} norm given by (3.10), (in other words, the H^{m+n-1} norm of the gradient). On the other hand we have

$$\int_M |(D\eta^T)^{-1} \nabla f_{m,n}|^2 dx dy \geq K_\eta \|\nabla f_{m,n}\|_{L^2}^2$$

where K_η is the infimum over M of the eigenvalues of $G_\eta = (D\eta^T D\eta)^{-1}$.

Next, consider the boundary term in (3.12) given by

$$\int_{\partial M} f_{m,n} \partial_x^m \partial_y^n (\partial_y g - |\partial_x \eta|^2 \partial_y f + \langle \partial_x \eta, \partial_y \eta \rangle \partial_x f) dx,$$

where we use the convention here and for the rest of the paper that $\int_{\partial M} h dx = \int_{S^1} h(x, L) dx - \int_{S^1} h(x, 0) dx$. Since $f|_{\partial M}$ is constant, we know that $f_{m,0}|_{\partial M} = 0$ so that this term vanishes if $n = 0$. If $n \geq 1$ then we can use the equation $\Delta g = \operatorname{div} (G_\eta \nabla f)$ to simplify

$$\partial_y (\partial_y g - |\partial_x \eta|^2 \partial_y f + \langle \partial_x \eta, \partial_y \eta \rangle \partial_x f) = -\partial_x (\partial_x g - |\partial_y \eta|^2 \partial_x f + \langle \partial_x \eta, \partial_y \eta \rangle \partial_y f) \quad (3.13)$$

so that the boundary term becomes the last term of (3.9) after an integration by parts in x .

² Here we agree to the convention that the boundary integral is zero if $n = 0$.

We will need to estimate a number of boundary terms of the form appearing in equation (3.9). The following lemma simplifies many of the calculations and will be used repeatedly.

Lemma 3.4 *Let $M = S^1 \times [0, L]$, and for any real function h on M , denote the oriented boundary integral by*

$$\int_{\partial M} h dx = \int_{S^1} h(x, L) dx - \int_{S^1} h(x, 0) dx.$$

For any H^1 functions f and g on M , we have

$$\int_{\partial M} f \partial_x g dx \leq \|\nabla f\|_{L^2} \|\nabla g\|_{L^2}.$$

Proof A straightforward computation gives

$$\begin{aligned} \int_{\partial M} f \partial_x g dx &= \int_{S^1} \int_0^L \frac{\partial}{\partial y} (f(x, y) \partial_x g(x, y)) dy dx \\ &= \int_M \partial_y f \partial_x g dx dy - \int_M \partial_x f \partial_y g dx dy \\ &= \left\langle (\partial_y f, -\partial_x f), (\partial_x g, \partial_y g) \right\rangle_{L^2} \leq \|\nabla f\|_{L^2} \|\nabla g\|_{L^2}. \end{aligned}$$

Before we proceed further with the proof of Claim (3.5) in full generality, let us illustrate the basic idea with a simple explicit example.

Example 3.5 For a positive constant ω , consider the diffeomorphism

$$\eta(x, y) = (x + \omega y, y).$$

This is a shear flow, and the function $\eta(t, x, y) = (x + t\omega y, y)$ is a solution of the inviscid Euler equation with steady velocity field $u(x, y) = \omega y e_x$. The matrix G_η is given by

$$G_\eta = (D\eta^T D\eta)^{-1} = \begin{pmatrix} 1 + \omega^2 & -\omega \\ -\omega & 1 \end{pmatrix}.$$

Consider the H^2 norm on vector fields, corresponding to the \dot{H}^3 norm on stream functions. (This is the first interesting case, as H^1 on vector fields has an accidental cancellation³ and L^2 on vector fields is the weak case already discussed.) The inner product defined by (3.6) reduces to

$$\langle\langle f, g \rangle\rangle_3 = B_0 \langle \nabla f_{yy}, \nabla g_{yy} \rangle_{L^2} + B_1 \langle \nabla f_{xy}, \nabla g_{xy} \rangle_{L^2} + B_2 \langle \nabla f_{xx}, \nabla g_{xx} \rangle_{L^2}. \quad (3.14)$$

The norm defined by (3.14) is equivalent to the \dot{H}^3 norm via the usual interpolation inequalities. Using the proof of Proposition 3.3, formula (3.14) becomes

$$\begin{aligned} \langle\langle f, g \rangle\rangle_3 &= B_0 \|G_\eta \nabla f_{yy}\|_{L^2}^2 + B_1 \|G_\eta \nabla f_{xy}\|_{L^2}^2 + B_2 \|G_\eta \nabla f_{xx}\|_{L^2}^2 \\ &\quad + B_0 \int_{\partial M} f_{xyy} (g_{xy} - (1 + \omega^2) f_{xy} + \omega f_{yy}) dx \\ &\quad + B_1 \int_{\partial M} f_{xxy} (g_{xx} - (1 + \omega^2) f_{xx} + \omega f_{xy}) dx \\ &\geq B_0 K_\eta \|\nabla f_{yy}\|_{L^2}^2 + B_1 K_\eta \|\nabla f_{xy}\|_{L^2}^2 + B_2 K_\eta \|\nabla f_{xx}\|_{L^2}^2 \\ &\quad + B_0 \int_{\partial M} f_{xyy} (g_{xy} - (1 + \omega^2) f_{xy}) dx \end{aligned} \quad (3.15)$$

after integrating by parts, an example of the accidental cancellation mentioned above. Here we use K_η to estimate the lowest eigenvalue of G_η , which is the reciprocal of the largest eigenvalue since G_η has determinant one.

We estimate the boundary term as follows, using Lemma 3.4 to get the following prototype of Proposition 4.1:

$$\begin{aligned} \int_{\partial M} f_{xyy} (g_{xy} - (1 + \omega^2) f_{xy}) dx &= \langle J \nabla f_{yy}, \nabla g_{xy} - (1 + \omega^2) \nabla f_{xy} \rangle \\ &\geq -\|\nabla f_{yy}\|_{L^2} \|\nabla g_{xy}\|_{L^2} - (1 + \omega^2) \|\nabla f_{yy}\|_{L^2} \|\nabla f_{xy}\|_{L^2}, \end{aligned} \quad (3.16)$$

³ See Lemma 4.2.

Now as a prototype of Lemma 4.2, we get an upper bound for $\|\nabla g_{xy}\|_{L^2}$ using formulas (3.11)–(3.12) with f replaced by g to get

$$\begin{aligned} \|\nabla g_{xy}\|_{L^2}^2 &= \int_M \langle \nabla g_{xy}, \partial_x \partial_y (G_\eta \nabla f) \rangle dx dy \\ &\quad + \int_{\partial M} g_{xxy} \partial_x (g_x - |\partial_y \eta|^2 \partial_x f) + \langle \partial_x \eta, \partial_y \eta \rangle \partial_y f dx \\ &= \int_M \langle \nabla g_{xy}, G_\eta \nabla f_{xy} \rangle dx dy + \omega \int_{\partial M} g_{xxy} f_{xy} dx \\ &\leq (1 + \omega)^2 \|\nabla g_{xy}\|_{L^2} \|\nabla f_{xy}\|_{L^2}, \end{aligned}$$

using Lemma 3.4 and the fact that the largest eigenvalue of G_η satisfies $1 + \omega^2 < \lambda < 1 + \omega + \omega^2$. Formula (3.16) then becomes

$$\begin{aligned} \int_{\partial M} f_{xyy} (g_{xy} - (1 + \omega^2) f_{xy}) dx &\geq -2(1 + \omega)^2 \|\nabla f_{xy}\|_{L^2} \|\nabla f_{yy}\|_{L^2} \\ &\geq -\varepsilon (1 + \omega)^2 \|\nabla f_{yy}\|^2 - \frac{(1 + \omega)^2}{\varepsilon} \|\nabla f_{xy}\|^2, \end{aligned}$$

for any $\varepsilon > 0$.

Formula (3.15) now becomes

$$\langle\langle f, g \rangle\rangle_3 \geq KB_0 \|\nabla f_{yy}\|_{L^2}^2 + KB_1 \|\nabla f_{xy}\|^2 + KB_2 \|\nabla f_{xx}\|_{L^2}^2 - \frac{B_0 \varepsilon}{K} \|\nabla f_{yy}\|_{L^2} - \frac{B_0}{\varepsilon K} \|\nabla f_{xy}\|_{L^2}^2,$$

where $K = (1 + \omega)^{-2}$ is a simple lower bound for K_η .

Now choose $\varepsilon = K^2/2$ with $B_0 = B_2 = 1$ and $B_1 = 3/K^4$. Then we end up with

$$\langle\langle f, g \rangle\rangle_3 \geq \frac{K}{3} \|f\|_3^2.$$

This gives formula (3.7) for $s = 2$, where the lower-order terms disappeared because G_η has constant coefficients. This is the main idea of the proof we give in the next section.

4 Proof of Theorem 1: Estimates at the Boundary

Now we estimate the boundary terms in Proposition 3.3 in terms of norms on the entire space M , using the fundamental Lemma 3.4.

Proposition 4.1 *Let $\eta \in \mathcal{D}_\mu(M)$. If $f \in \mathcal{F}^{s+1}(M)$ and $g = \Lambda f$ as in Lemma 3.2, then given any $m \geq 0$ and $n \geq 1$, the boundary terms in (3.9) can be estimated by*

$$\int_{\partial M} f_{m+1,n} g_{m+1,n-1} dx \leq \|\nabla f_{m,n}\|_{L^2} \|\nabla g_{m+1,n-1}\|_{L^2} \quad (n > 1) \quad (4.1)$$

$$\begin{aligned} \int_{\partial M} f_{m+1,n} \partial_x^m \partial_y^{n-1} (|\partial_y \eta|^2 \partial_x f) dx &\leq \|D\eta\|_{L^\infty}^2 \|\nabla f_{m,n}\|_{L^2} \|\nabla f_{m+1,n-1}\|_{L^2} \\ &\quad + C \|\eta\|_{C^{m+n+1}}^2 \|\nabla f_{m,n}\|_{L^2} \|f\|_{\dot{H}^{m+n}} \end{aligned} \quad (4.2)$$

$$\int_{\partial M} f_{m+1,n} \partial_x^m \partial_y^{n-1} (\langle \partial_x \eta, \partial_y \eta \rangle \partial_y f) dx \leq C \|\eta\|_{C^{m+n+1}}^2 \|\nabla f_{m,n}\|_{L^2} \|f\|_{\dot{H}^{m+n}} \quad (4.3)$$

where $C > 0$ is independent of η and the \dot{H}^{m+n} norm is defined in (3.10).

Proof Inequality (4.1) follows at once from Lemma 3.4. To estimate (4.2) we use Lemma 3.4 and the Leibniz rule to get

$$\begin{aligned} \int_{\partial M} f_{m+1,n} \partial_x^m \partial_y^{n-1} (|\partial_y \eta|^2 \partial_x f) dx &\leq \|\nabla f_{m,n}\|_{L^2} \|\nabla \partial_x^m \partial_y^{n-1} (|\partial_y \eta|^2 \partial_x f)\|_{L^2} \\ &\leq \|D\eta\|_{L^\infty}^2 \|\nabla f_{m+1,n-1}\|_{L^2} \|\nabla f_{m,n}\|_{L^2} + C \|\eta\|_{C^{m+n+1}}^2 \|f\|_{\dot{H}^{m+n}} \|\nabla f_{m,n}\|_{L^2} \end{aligned}$$

For (4.3) we use a trick on the highest-order term to improve over Lemma 3.4:

$$\int_{\partial M} f_{m+1,n} \partial_x^m \partial_y^{n-1} (\langle \partial_x \eta, \partial_y \eta \rangle \partial_y f) dx \leq \int_{\partial M} f_{m+1,n} \langle \partial_x \eta, \partial_y \eta \rangle f_{m,n} dx + C \|\eta\|_{C^{m+n+1}}^2 \|f\|_{\dot{H}^{m+n}} \|\nabla f_{m,n}\|_{L^2}$$

Now integrating the first term on the right by parts and estimating as in the proof of Lemma 3.4, we obtain

$$\begin{aligned} \int_{\partial M} f_{m+1,n} \langle \partial_x \eta, \partial_y \eta \rangle f_{m,n} dx &= \frac{1}{2} \int_{\partial M} \langle \partial_x \eta, \partial_y \eta \rangle \partial_x (f_{m,n}^2) dx = -\frac{1}{2} \int_{\partial M} \partial_x \langle \partial_x \eta, \partial_y \eta \rangle f_{m,n}^2 dx \\ &\lesssim \|\eta\|_{C^2}^2 \|f_{m,n}\|_{L^2} \|\nabla f_{m,n}\|_{L^2} \lesssim \|\eta\|_{C^2}^2 \|f\|_{\dot{H}^{m+n}} \|\nabla f_{m,n}\|_{L^2}, \end{aligned}$$

and thus this term folds into our previous term.

To understand the term (4.1), we want an upper bound for $\|\nabla g_{m,n}\|_{L^2}$ in terms of $\|\nabla f_{m,n}\|_{L^2}$. When n is small this works as in Example 3.5; when n is large we need to do the estimate recursively by replacing y derivatives with x derivatives using Δg , an extra complication.

Lemma 4.2 *Let η , f and $g = \Lambda f$ be as in Proposition 4.1, with $m \geq 1$ an integer. For any integer $n > 1$ we have*

$$\begin{aligned} \|\nabla g_{m,n}\|_{L^2} &\leq \|\nabla g_{m+1,n-1}\|_{L^2} + \|D\eta\|_{L^\infty}^2 \|\nabla f_{m+1,n-1}\|_{L^2} + \|D\eta\|_{L^\infty}^2 \|\nabla f_{m,n}\|_{L^2} \\ &\quad + C \|\eta\|_{C^{m+n+1}}^2 \|f\|_{\dot{H}^{m+n}} \end{aligned} \quad (4.4)$$

while for $n = 0$ or $n = 1$ we have

$$\|\nabla g_{m,n}\|_{L^2} \leq \|D\eta\|_{L^\infty}^2 \|\nabla f_{m,n}\|_{L^2} + C \|\eta\|_{C^{m+n+1}}^2 \|f\|_{\dot{H}^{m+n}} \quad (4.5)$$

where C is a constant depending on m and n but not on η .

Proof Integrating by parts as in (3.12), we have

$$\begin{aligned} \|\nabla g_{m,n}\|_{L^2}^2 &= \int_M \operatorname{div} (g_{m,n} \nabla g_{m,n}) dx dy - \int_M g_{m,n} \Delta g_{m,n} dx dy \\ &= \int_{\partial M} g_{m,n} \partial_x^m \partial_y^n (\langle \nabla g - G_\eta \nabla f, \nu \rangle) dx + \langle \nabla g_{m,n}, \partial_x^m \partial_y^n G_\eta \nabla f \rangle_{L^2}. \end{aligned} \quad (4.6)$$

We first consider the case when $m \geq 1$ and $n > 1$. Since $\langle G_\eta \nabla f, \partial_y \rangle = -\langle \partial_x \eta, \partial_y \eta \rangle \partial_x f + |\partial_x \eta|^2 \partial_y f$ from (3.13) we get

$$\partial_y (\partial_y g - G_\eta \nabla f, \partial_y) = -\partial_x (\partial_x g - |\partial_y \eta|^2 \partial_x f + \langle \partial_x \eta, \partial_y \eta \rangle \partial_y f)$$

Using this identity and integrating by parts in x the right hand side of (4.6) becomes

$$\begin{aligned} &= \int_{\partial M} g_{m+1,n} \partial_x^m \partial_y^{n-1} (\partial_x g - |\partial_y \eta|^2 \partial_x f + \langle \partial_x \eta, \partial_y \eta \rangle \partial_y f) dx + \langle \nabla g_{m,n}, \partial_x^m \partial_y^n G_\eta \nabla f \rangle_{L^2} \\ &= \int_{\partial M} g_{m+1,n} (g_{m+1,n-1} - |\partial_y \eta|^2 f_{m+1,n-1} + \langle \partial_x \eta, \partial_y \eta \rangle f_{m,n}) dx \\ &\quad + \sum_{0 < k+l < m+n} \int_{\partial M} \alpha_{kl} g_{m+1,n} f_{k,l} dx + \langle \nabla g_{m,n}, \partial_x^m \partial_y^n G_\eta \nabla f \rangle_{L^2} \\ &= I + II + III \end{aligned}$$

where α_{ij} are functions depending on the derivatives up to order $m+n-1$ of $|\partial_y \eta|^2$ and $\langle \partial_x \eta, \partial_y \eta \rangle$ and the binomial coefficients. Using Lemma 3.4 and (3.10) we have

$$\begin{aligned} |I| &\leq \|\nabla g_{m,n}\|_{L^2} \left(\|\nabla g_{m+1,n-1}\|_{L^2} + \|\nabla (|\partial_y \eta|^2 f_{m+1,n-1})\|_{L^2} + \|\nabla (\langle \partial_x \eta, \partial_y \eta \rangle f_{m,n})\|_{L^2} \right) \\ &\leq \|\nabla g_{m,n}\|_{L^2} \left(\|\nabla g_{m+1,n-1}\|_{L^2} + \|D\eta\|_{L^\infty}^2 (\|\nabla f_{m+1,n-1}\|_{L^2} + \|\nabla f_{m,n}\|_{L^2}) + \|\eta\|_{C^2}^2 \|f\|_{\dot{H}^{m+n}} \right) \end{aligned} \quad (4.7)$$

and similarly

$$|II| \lesssim \sum_{0 < k+l < m+n} \|\nabla g_{m,n}\|_{L^2} \|\nabla(\alpha_{kl} f_{k,l})\|_{L^2} \lesssim \|\eta\|_{C^{m+n+1}}^2 \|\nabla g_{m,n}\|_{L^2} \|f\|_{\dot{H}^{m+n}}. \quad (4.8)$$

Using the Cauchy-Schwarz inequality and the Leibniz rule

$$\begin{aligned} |III| &\leq \|\nabla g_{m,n}\|_{L^2} \|\partial_x^m \partial_y^n G_\eta \nabla f\|_{L^2} \\ &\leq \|D\eta\|_{L^\infty}^2 \|\nabla g_{m,n}\|_{L^2} \|\nabla f_{m,n}\|_{L^2} + C \|\nabla g_{m,n}\|_{L^2} \|\eta\|_{C^{m+n+1}}^2 \|f\|_{\dot{H}^{m+n}}. \end{aligned} \quad (4.9)$$

Combining (4.7), (4.8) and (4.9) we obtain (4.4), as desired.

Next, if $m \geq 1$ and $n = 0$ then the boundary term in (4.6) vanishes since $f|_{\partial M}$ and $g|_{\partial M}$ are constant, and we have

$$\begin{aligned} \|\nabla g_{m,0}\|_{L^2}^2 &\leq \|\nabla g_{m,0}\|_{L^2} \|\partial_x^m G_\eta \nabla f\|_{L^2} \\ &\leq \|D\eta\|_{L^\infty}^2 \|\nabla g_{m,0}\|_{L^2} \|\nabla f_{m,0}\|_{L^2} + C \|\nabla g_{m,0}\|_{L^2} \|\eta\|_{C^{m+1}}^2 \|f\|_{\dot{H}^m}. \end{aligned} \quad (4.10)$$

Finally, if $m \geq 1$ and $n = 1$ we use a trick to do a little better than (4.4). Integrating by parts in (4.6) as before we have

$$\begin{aligned} \|\nabla g_{m,1}\|_{L^2}^2 &= \int_{\partial M} g_{m+1,1} \partial_x^{m+1} g \, dx - \int_{\partial M} g_{m+1,1} \partial_x^m (|\partial_y \eta|^2 \partial_x f) \, dx \\ &\quad + \int_{\partial M} g_{m+1,1} \partial_x^m (\langle \partial_x \eta, \partial_y \eta \rangle \partial_y f) \, dx + \langle \nabla g_{m,1}, \partial_x^m \partial_y (G_\eta \nabla f) \rangle_{L^2} \end{aligned}$$

The first two terms on the right hand side drop out since $g|_{\partial M}$ and $f|_{\partial M}$ are both constant. The remaining terms can be estimated using Lemma 3.4 and the Cauchy-Schwarz inequality as before to get

$$\|\nabla g_{m,1}\|_{L^2}^2 \leq \|D\eta\|_{L^\infty}^2 \|\nabla g_{m,1}\|_{L^2} \|\nabla f_{m,1}\|_{L^2} + C \|\nabla g_{m,1}\|_{L^2} \|\eta\|_{C^{m+2}}^2 \|f\|_{\dot{H}^{m+1}} \quad (4.11)$$

where we used the homogeneous norm (3.10).

Our next task is to eliminate all g -terms on the right side of the basic inequality (4.4) using a simple recursive formula.

Proposition 4.3 *Let η , f and $g = \Lambda f$ be as in Lemma 4.2. For any $m \geq 1$ and $n \geq 0$ we have*

$$\|\nabla g_{m,n}\|_{L^2} \leq \|D\eta\|_{L^\infty}^2 \|\nabla f_{m,n}\|_{L^2} + 2\|D\eta\|_{L^\infty}^2 \sum_{k=1}^{n-1} \|\nabla f_{m+n-k,k}\|_{L^2} + C \|\eta\|_{C^{m+n+1}}^2 \|f\|_{\dot{H}^{m+n}} \quad (4.12)$$

for some constant C independent of η .

Proof Adding and subtracting terms and using inequalities (4.4) and (4.5) we have

$$\begin{aligned} \|\nabla g_{m,n}\|_{L^2} &= \|\nabla g_{m+n-1,1}\|_{L^2} + \sum_{k=1}^{n-1} \left(\|\nabla g_{m+n-k-1,k+1}\|_{L^2} - \|\nabla g_{m+n-k,k}\|_{L^2} \right) \\ &\leq \|D\eta\|_{L^\infty}^2 \|\nabla f_{m+n-1,1}\|_{L^2} + \|D\eta\|_{L^\infty}^2 \sum_{k=1}^{n-1} \left(\|\nabla f_{m+n-k-1,k+1}\|_{L^2} + \|\nabla f_{m+n-k,k}\|_{L^2} \right) \\ &\quad + C \|\eta\|_{C^{m+n+1}}^2 \|f\|_{\dot{H}^{m+n}}, \end{aligned}$$

and (4.12) follows.

Given an integer $s \geq 0$ and any positive numbers B_0, \dots, B_s define a semi-inner product on the space of stream functions (3.3) on M by

$$\langle\langle f, g \rangle\rangle_{s+1} = \sum_{j=0}^s B_j \langle \partial_x^j \partial_y^{s-j} \nabla f, \partial_x^j \partial_y^{s-j} \nabla g \rangle_{L^2} \quad (4.13)$$

and the associated seminorm by $\|f\|_{s+1} = \langle\langle f, f \rangle\rangle_{s+1}^{1/2}$. Interpolation inequalities show that this is actually a norm on $\mathcal{F}^{s+1}(M)$. Our goal is to show, using Propositions 4.1 and 4.3, that the B_j can be chosen so that A_t is positive-definite up to highest-order terms in the inner product (4.13).

Proposition 4.4 *Let $\eta(t)$ be a smooth curve in $\mathcal{D}_\mu(M)$ on $[0, T]$. Let f be a smooth function with $f|_{\partial M}$ constant, and let $g(t) = \Lambda_t f$. Given $s \geq 1$ there exist positive coefficients B_0, \dots, B_s depending on η but independent of t such that for sufficiently small $\epsilon > 0$ we have*

$$\langle\langle f, g \rangle\rangle_{s+1} \geq K \|f\|_{s+1}^2 - C \|f\|_{s+1} \|f\|_{\dot{H}^s} \quad (4.14)$$

where $K > 0$ and $C > 0$ are constants depending on ϵ and s ; in addition K depends on the $L^\infty C_x^1$ -norm and C depends on the $L_t^\infty C_x^{s+1}$ -norm⁴ of η .

Proof From Proposition 3.3 for any $t \geq 0$ and any integers $m \geq 0$ and $n \geq 0$ with $m + n = s$ we have

$$\begin{aligned} \langle\langle \nabla f_{mn}, \nabla g_{mn}(t) \rangle\rangle_{L^2} &\geq K_\eta \|\nabla f_{m,n}\|_{L^2}^2 - C \|\eta\|_{L_t^\infty C_x^{s+1}}^2 \|\nabla f_{m,n}\|_{L^2} \|f\|_{\dot{H}^s} \\ &\quad + \int_{\partial M} f_{m+1,n} \partial_x^m \partial_y^{n-1} (\partial_x g - |\partial_y \eta|^2 \partial_x f + \langle \partial_x \eta, \partial_y \eta \rangle \partial_y f) dx \end{aligned}$$

for some constant C independent of η . Note that by convention⁵ the integral over the boundary vanishes if $n = 0$ and, furthermore, the first term of the integral (corresponding to the factor $\partial_x g$) also vanishes if $n = 1$ (since $g|_{\partial M}$ is constant by assumption). Therefore, using Proposition 4.1, we can now estimate the above expression from below by (using various constants, all of which we denote by C)

$$\begin{aligned} \langle \nabla f_{m,n}, \nabla g_{m,n} \rangle_{L^2} &\geq K_\eta \|\nabla f_{m,n}\|_{L^2}^2 - \|\nabla f_{m,n}\|_{L^2} \left(\|\nabla g_{m+1,n-1}\|_{L^2} + \|D\eta\|_{L_{[0,T] \times M}^\infty}^2 \|\nabla f_{m+1,n-1}\|_{L^2} \right) \\ &\quad - C \|\eta\|_{L_t^\infty C_x^{s+1}}^2 \|f\|_{\dot{H}^s} \|\nabla f_{m,n}\|_{L^2} \end{aligned}$$

and, with the help of Proposition 4.3 and rearranging and combining like terms, estimate it even further by

$$\begin{aligned} \langle \nabla f_{m,n}, \nabla g_{m,n} \rangle_{L^2} &\geq K_\eta \|\nabla f_{m,n}\|_{L^2}^2 - 2 \|D\eta\|_{L_{[0,T] \times M}^\infty}^2 \|\nabla f_{m,n}\|_{L^2} \sum_{k=1}^{n-1} \|\nabla f_{m+n-k,k}\|_{L^2} \\ &\quad - C \|\eta\|_{L_t^\infty C_x^{s+1}}^2 \|\nabla f_{m,n}\|_{L^2} \|f\|_{\dot{H}^s} \\ &\geq \left(K_\eta - (s-1)\epsilon \|D\eta\|_{L_{[0,T] \times M}^\infty}^2 \right) \|\nabla f_{m,n}\|_{L^2}^2 - \frac{1}{\epsilon} \|D\eta\|_{L_{[0,T] \times M}^\infty}^2 \sum_{k=1}^{n-1} \|\nabla f_{m+n-k,k}\|_{L^2}^2 \\ &\quad - C \|\eta\|_{L_t^\infty C_x^{s+1}}^2 \|\nabla f_{m,n}\|_{L^2} \|f\|_{\dot{H}^s} \end{aligned}$$

for any positive ϵ .

Setting

$$C_\eta = C \|\eta\|_{L_t^\infty C_x^{s+1}}^2 \quad (4.15)$$

$$Q_\epsilon = \frac{1}{\epsilon} \|D\eta\|_{L_{[0,T] \times M}^\infty}^2 \quad (4.16)$$

$$K_\epsilon = \inf_{0 \leq \tau \leq t} K_\eta - \frac{(s-1)}{2} \epsilon \|\eta\|_{L_t^\infty C_x^{s+1}}^2 \quad (4.17)$$

and choosing

$$0 < \epsilon < \frac{\inf_{0 \leq \tau \leq t} K_\eta}{(s-1) \|\eta\|_{L_{[0,T] \times M}^\infty}^2} \quad (4.18)$$

we therefore obtain

$$\langle \nabla f_{m,n}, \nabla g_{m,n} \rangle_{L^2} \geq K_\epsilon \|\nabla f_{m,n}\|_{L^2}^2 - Q_\epsilon \sum_{k=1}^{n-1} \|\nabla f_{m+n-k,k}\|_{L^2}^2 - C_\eta \|\nabla f_{m,n}\|_{L^2} \|f\|_{\dot{H}^{m+n}} \quad (4.19)$$

for any integers $m \geq 0$ and $n \geq 0$.

⁴ That is, $\|\varphi\|_{L_t^\infty C_x^k} = \sup_{0 \leq \tau \leq t} \|\varphi(\tau)\|_{C^k}$.

⁵ See the footnote to Proposition 3.3.

Next, let $s \geq 1$. In terms of the inner product (4.13) consider

$$\begin{aligned} \langle\langle f, g \rangle\rangle_{s+1} &= \sum_{j=0}^s B_j \langle \partial_x^j \partial_y^{s-j} \nabla f, \partial_x^j \partial_y^{s-j} \nabla g \rangle_{L^2} = \sum_{j=0}^s B_j \langle \nabla f_{j,s-j}, \nabla g_{j,s-j} \rangle_{L^2} \\ &\geq \sum_{j=0}^s B_j \left(K_\epsilon \|\nabla f_{j,s-j}\|_{L^2}^2 - Q_\epsilon \sum_{k=1}^{s-j-1} \|\nabla f_{s-k,k}\|_{L^2}^2 - C_\eta \|\nabla f_{j,s-j}\|_{L^2} \|f\|_{\dot{H}^s} \right) \\ &\geq B_0 K_\epsilon \|\nabla f_{0,s}\|_{L^2}^2 + B_s K_\epsilon \|\nabla f_{s,0}\|_{L^2}^2 + \sum_{k=1}^{s-1} \left(K_\epsilon B_k - Q_\epsilon \sum_{j=0}^{k-1} B_j \right) \|\nabla f_{k,s-k}\|_{L^2}^2 \\ &\quad - C_{\eta,B}^s \|f\|_{s+1} \|f\|_{\dot{H}^s} \end{aligned}$$

where $C_{\eta,B}^s = C_\eta (s+1)^{1/2} \max_{0 \leq j \leq s} \sqrt{B_j}$. Now, for any $1 \leq k \leq s-1$ pick

$$B_k = \frac{2}{K_\epsilon} Q_\epsilon \sum_{j=0}^{k-1} B_j \quad (4.20)$$

and set $B_0 = B_s = 1$. Note that the solution of the recurrence equation in (4.20) is easily found to be $B_k = (1 + 2Q_\epsilon/K_\epsilon)^{k-1} 2Q_\epsilon/K_\epsilon$ where Q_ϵ, K_ϵ and $\epsilon > 0$ are given by (4.16), (4.17) and (4.18). Combining these we now obtain

$$\begin{aligned} \langle\langle f, g \rangle\rangle_{s+1} &\geq \frac{1}{2} K_\epsilon \left(\|\nabla f_{0,s}\|_{L^2}^2 + \sum_{k=1}^{s-1} B_k \|\nabla f_{k,s-k}\|_{L^2}^2 + \|\nabla f_{s,0}\|_{L^2}^2 \right) - C_{\eta,B}^s \|f\|_{s+1} \|f\|_{\dot{H}^s} \\ &= \frac{1}{2} K_\epsilon \|f\|_{s+1}^2 - C_{\eta,B}^s \|f\|_{s+1} \|f\|_{\dot{H}^s} \end{aligned}$$

which is the desired estimate. Finally note that since the determinant of G_η is one, the infimum K_η is the reciprocal of $\|D\eta\|_{L^\infty}^2$, and thus the constant K appearing in (4.14) is the reciprocal of $\|D\eta\|_{L^\infty_{[0,T] \times M}^2}$, as claimed.

We can now address the Claim (3.5).

Proposition 4.5 *Let $M = S^1 \times [0, L]$ and let $\eta(t)$ be a smooth curve of area-preserving diffeomorphisms $\mathcal{D}_\mu(M)$. Given any $t > 0$ the operator $\widehat{\Omega}_t = \int_0^t \Lambda_\tau^{-1} d\tau$ defined in (3.5) on the space $\mathcal{F}^{s+1}(M)$ of stream functions to itself is invertible.*

Proof For any $0 \leq \tau \leq t$ applying Proposition 4.4 to $f = \Lambda_\tau^{-1} g$ we obtain

$$\langle\langle g, \Lambda_\tau^{-1} g \rangle\rangle_{s+1} \geq K \|\Lambda_\tau^{-1} g\|_{s+1}^2 - C \|\Lambda_\tau^{-1} g\|_{s+1} \|\Lambda_\tau^{-1} g\|_{\dot{H}^s}. \quad (4.21)$$

Proposition 4.3 implies that Λ_t is a bounded operator in the topology defined by (4.13) (or in any Sobolev norm on $\mathcal{F}^{s+1}(M)$), so that

$$\|g\|_{s+1} = \|\Lambda_\tau \Lambda_\tau^{-1} g\|_{s+1} \leq \|\Lambda_\tau\|_{s+1} \|\Lambda_\tau^{-1} g\|_{s+1}.$$

The open mapping theorem then implies that Λ_τ^{-1} is also bounded in the same topology. The inequality (4.21) then becomes

$$\langle\langle g, \Lambda_\tau^{-1} g \rangle\rangle_{s+1} \geq K N_1^{-2} \|g\|_{s+1}^2 - C N_2^{-1} N_3^{-1} \|g\|_{s+1} \|g\|_{\dot{H}^s},$$

where

$$N_1 = \sup_{0 \leq \tau \leq t} \|\Lambda_\tau\|_{s+1}, \quad N_2 = \inf_{0 \leq \tau \leq t} \|\Lambda_\tau\|_{s+1} \quad \text{and} \quad N_3 = \inf_{0 \leq \tau \leq t} \|\Lambda_\tau\|_{\dot{H}^s}.$$

Integrating both sides of (4.21) over $[0, t]$ and using Cauchy-Schwarz we get

$$\|\widehat{\Omega}_t g\|_{s+1} \geq K t N_1^{-2} \|g\|_{s+1} - C t N_2^{-1} N_3^{-1} \|g\|_{\dot{H}^s}.$$

It follows that $\widehat{\Omega}_t$ has closed range. By Lemma 1, $\widehat{\Omega}_t$ has trivial null-space and it follows that $\widehat{\Omega}_t$ is semi-Fredholm. Since the index of semi-Fredholm operators is constant under continuous perturbations, and since it is zero at $t = 0$, we conclude that the index is always zero. Therefore, $\widehat{\Omega}_t$ also has trivial cokernel and must be invertible on the space \mathcal{F}^{s+1} .

It now follows that given any smooth divergence-free vector field v_0 on M the corresponding operator Ω_t on $T_e\mathcal{D}_\mu$ is also invertible which, in light of Proposition 2.1, implies that $\Phi_t = D\eta_t(\Omega_t - \Gamma_t)$ is the sum of an invertible operator and a compact operator. We conclude that Φ_t is a Fredholm operator of index zero. This concludes the proof of Theorem 1.1 in the smooth case $v_0 \in T_e\mathcal{D}_\mu$. The H^s case follows by a perturbation argument as in [8] or [14] and will be omitted. The only important thing to note is that our leading-term estimates depend only on the C^1 norm $\|D\eta\|_{L^\infty}$, and thus when we approximate an $\eta \in H^s$ by an $\tilde{\eta} \in C^\infty$, the coefficients in the leading term can be made as close as we want to those we found above. Again we refer to [8] for details.

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