EULER-ARNOLD EQUATIONS AND TEICHMÜLLER THEORY

STEPHEN C. PRESTON AND PEARCE WASHABAUGH

Abstract. In this paper we prove that for $s > 3/2$, all H^s solutions of the Euler-Weil-Petersson equation, which describes geodesics on the universal Teichmüller space under the Weil-Petersson metric, will remain in H^s for all time. This extends the work of Escher-Kolev for strong Riemannian metrics to the borderline case of $H^{3/2}$ metrics. In addition we show that all H^s solutions of the Wunsch equation, a variation of the Constantin-Lax-Majda equation which also describes geodesics on the universal Teichmüller curve under the Velling-Kirillov metric, must blow up in finite time due to wave breaking, extending work of Castro-Córdoba and Bauer-Kolev-Preston. Finally we illustrate these phenomena numerically.

Contents

1. Introduction

Euler-Arnold equations are PDEs that describe the evolution of a velocity field for which the Lagrangian flow is a geodesic in a group of smooth diffeomorphisms of a manifold, for some choice of right-invariant Riemannian metric; see Arnold-Khesin [1]. In the one-dimensional case, we will consider the diffeomorphism group of the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. If the Riemannian metric is defined at the identity by

$$\langle u, u\rangle_r = \int_{S^1} u\Lambda^{2r} u \, d\theta, \tag{1}$$

where Λ^{2r} is a symmetric, positive pseudodifferential operator of order r, we call it a Sobolev H^r metric, and the Euler-Arnold equation is given by

$$m_t + um_\theta + 2mu_\theta = 0, \qquad m = \Lambda^{2r}u, \qquad u = u(t,\theta), \qquad u(0) = u_0 \in C^\infty(S^1). \tag{2}$$

Special cases include the Camassa-Holm equation when $r = 1$ and $\Lambda^2 = 1-\partial_\theta^2$, or the right-invariant Burgers' equation when $r = 0$ and $\Lambda^0 = 1$ [4]. One can also allow Λ^{2r} to be degenerate (nonnegative

Date: March 16, 2018.

2010 *Mathematics Subject Classification.* 35Q35, 53D25.

Key words and phrases. Euler-Weil-Petersson equation, global existence, universal Teichmüller space, Sobolev metrics of fractional order.

S. C. Preston gratefully acknowledges support from Simons Collaboration Grant no. 318969, and the hospitality of the organizers of the "Math on the Rocks" workshop in Grundsund, Sweden, where part of this research was conducted.

rather than positive); the best known example is when $r = 1$ and $\Lambda^2 = -\partial_\theta^2$, for which we get the Hunter-Saxton equation [12]. Here we are interested in the fractional order cases $r = \frac{1}{2}$ and $r = \frac{3}{2}$ (see Escher-Kolev [7]), which arise naturally in Teichmüller theory [10]. Both cases are critical in some sense, due to the Sobolev embedding being critical: for $r < \frac{1}{2}$ Lagrangian trajectories do not depend smoothly on initial conditions, while for $r > \frac{3}{2}$ conservation of energy is strong enough to ensure global existence [8]. In this paper we will show that all solutions for $r = \frac{1}{2}$ blow up in finite time while for $r = \frac{3}{2}$ all smooth solutions exist globally; previously only some solutions were known to blow up in the $r = \frac{1}{2}$ case [2] and smooth solutions were only known to stay in $H^{3/2}$ in the $r = \frac{3}{2}$ case [10].

Specifically the cases we are interested in are

- ($r = \frac{1}{2}$) the Wunsch equation [22],[2]: $\Lambda^1 = Hu_\theta$,
- and ($r = \frac{3}{2}$) the Euler-Weil-Petersson equation [10]: $\Lambda^3 = -H(u_{\theta\theta\theta} + u_\theta)$,

where H is the Hilbert transform defined for periodic functions by $H(e^{in\theta}) = -i \operatorname{sign} n e^{in\theta}$. The Wunsch equation is a special case of the modified Constantin-Lax-Majda equation [16] which models vorticity growth in an ideal fluid.

When paired with the flow equation

$$\frac{\partial \eta}{\partial t}(t,\theta) = u\big(t, \eta(t,\theta)\big), \qquad \eta(0,\theta) = \theta, \tag{3}$$

the Euler-Arnold equation (2) describes geodesics $\eta(t)$ of the right-invariant Riemannian metric defined at the identity element by (1) on the homogeneous space $\operatorname{Diff}(S^1)/G$. Here G is the group generated by the subalgebra $\ker \Lambda$ of length-zero directions: for the Euler-Weil-Petersson equation we have $G = \mathrm{PSL}_2(\mathbb{R})$, and for the Wunsch equation we have $G = \mathrm{Rot}(S^1) \cong S^1$.

The local existence result was obtained by Escher-Kolev [7], a strengthening of a result of Escher-Kolev-Wunsch [9].

Theorem 1 (Escher-Kolev)**.** *Suppose Λ^r is either $\Lambda^1 = Hu_\theta$ or $\Lambda^3 = -H(u_{\theta\theta\theta} + u_\theta)$. Then the system (2)–(3) is a smooth ODE for $\eta \in \operatorname{Diff}^s(S^1)/G$, for $s > \frac{3}{2}$ and $G = \mathrm{Rot}(S^1)$ or $G = \mathrm{PSL}_2(\mathbb{R})$, respectively. Hence for any $u_0 \in H^s(S^1)$, there is a unique solution $\eta\colon [0,T) \to \operatorname{Diff}^s(S^1)/G$ with $\eta(0) = \mathrm{id}$ and $\eta_t(0) = u_0$, with the map $u_0 \mapsto \eta(t)$ depending smoothly on u_0.*

Loss of smoothness in the equation (2) occurs due to the fact that composition required to get $u = \dot\eta \circ \eta^{-1}$ is not smooth in η; thus although the second-order equation for η (with u eliminated) is an ODE, the first-order equation (2) for u alone is *not* an ODE. This approach to the Euler equations was originally due to Ebin-Marsden [6]; for the Wunsch equation it was proved by Escher-Kolev-Wunsch [9] for large Sobolev indices, while for the Euler-Weil-Petersson equation it was proved by Escher-Kolev [7]. Castro-Córdoba [3] showed that if u_0 is initially odd, then solutions to the Wunsch equation blow up in finite time; the authors of [2] extended this result to some data without odd symmetry. For the Euler-Weil-Petersson equation, it was not known whether initially smooth data would remain smooth for all time. However Gay-Balmaz and Ratiu [10] interpreted the equation in $H^{3/2}$ as a *strong* Riemannian metric on a certain manifold and concluded that the velocity field u remains in $H^{3/2}(S^1)$ for all time. We complement this to obtain a uniform C^1 bound, which then by bootstrapping gives uniform bounds on all Sobolev norms H^s for $s > \frac{3}{2}$, and thus in particular we show that initially smooth solutions remain smooth.

The main theorems of this paper settle the global existence question for the degenerate $\dot H^r$ metrics corresponding to $r = \frac{1}{2}$ (the Wunsch equation) and $r = \frac{3}{2}$ (the Euler-Weil-Petersson equation).

Theorem 2. *Suppose $s > \frac{3}{2}$ and u_0 is an H^s velocity field on S^1 with mean zero (i.e., $u_0 \in H^s(S^1)/\mathbb{R}$). Then the solution $u(t)$ of the Wunsch equation with $u(0) = u_0$ blows up in finite time.*

Theorem 3. *Suppose $s > \frac{3}{2}$ and u_0 is an H^s velocity field on S^1, and that the Fourier series of u_0 has vanishing $n = 0$, $n = 1$, and $n = -1$ component; i.e., $u_0 \in H^s(S^1)/\mathfrak{sl}_2(\mathbb{R})$. Then the solution $u(t)$ of the Euler-Weil-Petersson equation with $u(0) = u_0$ remains in H^s for all time. In particular if u_0 is C^∞ then so is $u(t)$ for all $t > 0$.*

Additionally, Theorem 2 almost immediately gives us that every mean zero solution of the Constantin-Lax-Majda equation [5] blows up in finite time. Overall, these two Theorems mean that the case $r = \frac{3}{2}$ behaves the same as the cases for $r > \frac{3}{2}$, while the case $r = \frac{1}{2}$ behaves the same as for $r = 1$ (since all solutions of the Hunter-Saxton equation blow up in finite time [14]). We may conjecture that there is a critical value r_0 such that for $r > r_0$ all smooth mean-zero solutions remain smooth for all time, while for $r < r_0$ all smooth mean-zero solutions blow up in finite time. Our guess is that $r_0 = \frac{3}{2}$, but the current method does not prove this; furthermore we do not know what happens with geodesics for $\frac{1}{2} < r < 1$ or $1 < r < \frac{3}{2}$ even in the degenerate case.

Both equations arise naturally in the study of universal Teichmüller spaces. The Euler-Weil-Petersson equation was derived in [10] as the Euler-Arnold equation arising from the Weil-Petersson metric on the universal Teichmüller space. This geometry has been studied extensively by Takhtajan-Teo [18]; in particular they constructed the Hilbert manifold structure that makes Weil-Petersson a strong Hilbert metric (thus ensuring that geodesics exist globally). The Weil-Petersson geometry is well-known: the sectional curvature is strictly negative, and it is a Kähler manifold with almost complex structure given by the Hilbert transform. See Tromba [21] and Yamada [23] for further background on the Weil-Petersson metric on the universal Teichmüller space.

The Wunsch equation arises from the Riemannian metric $\langle u, u\rangle = \int_{S^1} uHu_\theta\,dx$, which is called the Velling-Kirillov metric and was proposed as a metric on the universal Teichmüller curve by Teo [19][20]. The Velling-Kirillov geometry was originally studied by Kirillov-Yur'ev [13]; although the sectional curvature is believed to be always positive, this is not yet proved. Furthermore the geometries are related in the sense that integrating the square of the symplectic form for the W-P geometry gives the symplectic form for the V-K geometry. Yet the properties of these geometries seem to be opposite in virtually every way: from Fredholmness of the exponential map [15][2] to the sectional curvature to the global properties of geodesics mentioned above.

The authors would like to thank Martin Bauer and Boris Kolev for suggesting the problem and useful discussions on the result. The first author would like to thank the organizers and participants of the "Math on the Rocks" shape analysis workshop in Grundsund, Sweden during July 2015, where parts of the proofs of several theorems were discovered.

2. Proof of the Main Theorems

2.1. Rewriting the Equations and Proof of Theorem 2.

Let us first sketch the blowup argument for the Wunsch equation from [2], which extended the argument of Castro-Córdoba [3]. The Wunsch equation is given for smooth mean-zero vector fields u on S^1 (identified with functions) by the formula

$$\omega_t + u\omega_\theta + 2u_\theta\omega = 0, \qquad \omega = Hu_\theta. \tag{4}$$

In terms of the Lagrangian flow η given by (3), we may rewrite this as

$$\partial_t\omega\big(t, \eta(t,\theta)\big) + 2\eta_{t\theta}(t,\theta)\omega\big(t,\eta(t,\theta)\big)/\eta_\theta(t,\theta) = 0$$

which leads to the conservation law

$$\eta_\theta(t,\theta)^2\omega\big(t,\eta(t,\theta)\big) = \omega_0(\theta).$$

Applying the Hilbert transform to both sides of (4) and using the following Hilbert transform identities (valid for mean-zero functions f):

$$H(Hf) = -f \qquad \text{and} \qquad 2H(fHf) = (Hf)^2 - f^2, \tag{5}$$

one obtains [2] an equation for $u_\theta = -H\omega$:

$$u_{t\theta} + uu_{\theta\theta} + u_\theta^2 = -F + \omega^2 \tag{6}$$

where the function F is a spatially nonlocal force given for each fixed time t by

$$F = -uu_{\theta\theta} - H(uHu_{\theta\theta}). \tag{7}$$

This function F is positive everywhere for any mean-zero function u: see Theorem 6.

In Lagrangian form, using the conservation law equation (6) becomes

$$\eta_{tt\theta}(t,\theta) = \frac{\omega_0(\theta)^2}{\eta_\theta(t,\theta)^3} - F\big(t, \eta(t,\theta)\big)\eta_\theta(t,\theta). \tag{8}$$

Now although we derive (8) for smooth solutions, we may observe that the equation (8) for η_θ is an ordinary differential equation which makes sense for $\eta \in H^s$ for $s > \frac{3}{2}$. In particular it is not hard to show that $F(t,\theta)$ given by (7) is in H^{s-1} as long as $u \in H^s$ for $s > \frac{3}{2}$, and in fact F is differentiable as a function of u since it depends only quadratically on u. As such the right side of (8) is a smooth function of $\eta_\theta \in H^{s-1}$ in the sense of Fréchet derivatives, and thus we obtain a smooth ODE on the space of positive H^{s-1} functions. We omit the details since the result is the same as that of Escher-Kolev [7] quoted above in Theorem 1.

If there is a point θ_0 such that $u'(\theta_0) < 0$ and $\omega_0(\theta_0) = 0$, then we will have $\eta_\theta(0,\theta_0) = 1$ and $\eta_{t\theta}(0,\theta_0) = u_\theta(0,\theta_0) < 0$. Since $F(t,\eta(t,\theta_0)) > 0$ for every t, we see that $\eta_{tt\theta}(t,\theta_0) < 0$ for all t, so that $\eta_\theta(t,\theta_0)$ must reach zero in finite time (which leads to $u_\theta \to -\infty$). Our proof that *all* solutions blow up consists of showing that this condition happens for *every* initial condition u_0 with $\omega_0 = Hu_0$.

Proof of Theorem 2. Note that $u_0 \in H^s$ for $s > \frac{3}{2}$, and thus u_0' is continuous. From the discussion above, the proof reduces to proving the following statement. Suppose $f\colon S^1 \to \mathbb{R}$ is a continuously differentiable function with mean zero, and let $g = Hf$. Then there is a point $\theta_0 \in S^1$ with $f'(\theta_0) < 0$ and $g'(\theta_0) = 0$.

Let p be the unique harmonic function in the unit disc $\mathbb{D}$ such that $p|_{S^1} = f$, and let q be its harmonic conjugate normalized so that $q|_{S^1} = g$. Then in polar coordinates we have the Cauchy-Riemann equations

$$rp_r(r,\theta) = q_\theta(r,\theta) \qquad \text{and} \qquad rq_r(r,\theta) = -p_\theta(r,\theta), \tag{9}$$

and we have $p(1,\theta) = f(\theta)$ and $q(1,\theta) = g(\theta)$.

Since q is harmonic, its maximum value within $\mathbb{D}$ occurs on the boundary S^1 at some point θ_0. The maximum of g occurs at the same point, so that $g'(\theta_0) = 0$. By the Hopf lemma, we have $q_r(1,\theta_0) > 0$, so equations (9) imply that $f'(\theta_0) = p_\theta(1,\theta_0) < 0$. □

Remark 4. This argument also works when the domain is $\mathbb{R}$ and the functions have suitable decay conditions imposed. It can thus be applied to demonstrate that *every* mean zero solution of the Constantin-Lax-Majda equation [5]

$$\omega_t - v_x\omega = 0, \ \ v_x = H\omega$$

blows up in finite time, using the same argument as in that paper via the explicit solution formula.

Now let us rewrite the Euler-Weil-Petersson equation to obtain the analogue of formula (6). Recall from the introduction that it is given explicitly by

$$\omega_t + u\omega_\theta + 2u_\theta\omega = 0, \qquad \omega = -Hu_{\theta\theta\theta} - Hu_\theta. \tag{10}$$

Proposition 5. *For a smooth velocity field u, the Euler-Weil-Petersson equation (10) is equivalent to the equation*

$$u_{t\theta} = H(uHu_{\theta\theta}) + H(1+\partial_\theta^2)^{-1}\big[2u_\theta Hu_\theta - u_{\theta\theta}Hu_{\theta\theta}\big], \tag{11}$$

In terms of the Lagrangian flow (3), equation (11) takes the form

$$\frac{\partial}{\partial t}u_\theta(t,\eta(t,\theta)) = -F(t,\eta(t,\theta)) + G(t,\eta(t,\theta)) \tag{12}$$

where F is defined by formula (7) and G is given by

$$G = H(1+\partial_\theta^2)^{-1}[2u_\theta H u_\theta - u_{\theta\theta}Hu_{\theta\theta}]. \tag{13}$$

Here the operator $(1+\partial_\theta^2)$ is restricted to the orthogonal complement of the span of $\{1, \sin\theta, \cos\theta\}$ so as to be invertible.

Proof. Equation (10) may be written

$$-H(1+\partial_\theta^2)u_{t\theta} = (1+\partial_\theta^2)(uHu_{\theta\theta}) - u_{\theta\theta}Hu_{\theta\theta} + 2u_\theta Hu_\theta,$$

using the product rule. We now solve for $u_{t\theta}$ by applying H to both sides and inverting $(1+\partial_\theta^2)$. To do this, we just need to check that the term $(2u_\theta Hu_\theta - u_{\theta\theta}Hu_{\theta\theta})$ is orthogonal to the subspace spanned by $\{1, \sin\theta, \cos\theta\}$. In fact this is true for every function fHf when f is 2π-periodic with mean zero, since the formulas (5) imply both that fHf has mean zero and that it has period π.

The only additional thing happening in equation (12) is the chain rule formula

$$\partial_t u_\theta(t,\eta(t,\theta)) = u_{t\theta}(t,\eta(t,\theta)) + u_{\theta\theta}(t,\eta(t,\theta))\eta_t(t,\theta) = (u_{t\theta}+uu_{\theta\theta})(t,\eta(t,\theta)).$$

□

Equation (12) may now be written in the form

$$\eta_{tt\theta}(t,\theta) = \big(G(t,\eta(t,\theta)) - F(t,\eta(t,\theta))\big)\eta_\theta(t,\theta) + \frac{\eta_{t\theta}(t,\theta)^2}{\eta_\theta(t,\theta)}. \tag{14}$$

As mentioned earlier, $u \mapsto F$ is a smooth function from H^s mean-zero functions u to positive H^{s-1} functions. It is also not difficult to prove that G given by (13) also takes H^s to H^{s-1}, and thus we see that (14) is a smooth second-order ODE in the space of positive H^{s-1} functions η_θ. Again we omit the details since the result is equivalent to the Escher-Kolev Theorem 1.

To prove Theorem 3, we want to show that $\|u_\theta\|_{L^\infty}$ remains bounded for all time, and by formula (12) it is sufficient to bound both $\|F\|_{L^\infty}$ and $\|G\|_{L^\infty}$. We will do this in the next Section.

2.2. **The Bounds on F and G.** In [2], it was shown that the function F given by (7) is positive for any mean-zero function $u\colon S^1 \to \mathbb{R}$. This is essential for proving blowup for the Wunsch equation.

Theorem 6 (Bauer-Kolev-Preston)**.** *Let $u\colon S^1 \to \mathbb{R}$ be a function with Fourier series $u(\theta) = \sum_{n\in\mathbb{Z}} c_n e^{in\theta}$ with $c_0 = 0$. If $\Lambda = H\partial_\theta$ so that $\Lambda(e^{in\theta}) = |n|e^{in\theta}$, and if $g_p = H(uH\Lambda^p u) + u\Lambda^p u$ for a positive number p, then for every $\theta \in S^1$ we have*

$$g_p(\theta) = 2\sum_{k=1}^{\infty}\left[k^p - (k-1)^p\right]|\phi_k(x)|^2, \quad \textit{where } \phi_k(\theta) = \sum_{m=k}^{\infty} c_m e^{im\theta} \tag{15}$$

In particular $F = -uu'' - H(uHu'')$ is positive at every point if u is not constant.

Another perspective on the positivity of F is discussed in Silvestre-Vicol [17]. There, while studying a slightly different version of the generalized Constantin-Lax-Majda equation over $\mathbb{R}$, they demonstrated that for any function u on $\mathbb{R}$ rather than on S^1, the function F defined by (7) can also be represented as

$$F(0) = \left\|\frac{u(x)-u(0)}{x}\right\|^2_{\dot{H}^{1/2}(\mathbb{R})}.$$

This insight into the structure of F helps explain the positivity result of the previous Theorem. There is a similar integral formula for functions on the circle, but we will not need it here. We would now like a pointwise bound for bound F in terms of $\|u\|^2_{\dot H^{3/2}}$.

Theorem 7. *Let $u : S^1 \to \mathbb{R}$ be a smooth function with Fourier coefficients c_n such that $c_0 = c_1 = c_{-1} = 0$, and let $F = -uu'' - H(uHu'')$. Then for every $\theta \in S^1$, we have*

$$F(\theta) \le \tfrac{1}{2\pi}\|u\|^2_{\dot H^{3/2}},$$

where

$$\|u\|^2_{\dot H^{3/2}(S^1)} = \int_{S^1} (Hu)(u''' + u')\,d\theta = 4\pi \sum_{n=2}^{\infty} (n^3 - n)|c_n|^2.$$

Proof. Without loss of generality we may assume that $\theta = 0$ for simplicity. Since $c_1 = 0$, we have $\phi_1(0) = \phi_2(0)$, and thus by equation (15) we have (using the Cauchy-Schwarz inequality)

$$\begin{aligned} F(0) &= \sum_{n=1}^{\infty} (2n-1)\Big|\sum_{m=n}^{\infty} c_m\Big|^2 \le \sum_{n=2}^{\infty} 2n \sum_{m=n}^{\infty} m(m+1)|c_m|^2 \sum_{m=n}^{\infty} \frac{1}{m(m+1)} \\ &\le \sum_{n=2}^{\infty} \frac{2n}{n} \sum_{m=n}^{\infty} m(m+1)|c_m|^2 \le 2\sum_{m=2}^{\infty}\sum_{n=2}^{m} m(m+1)|c_m|^2 = 2\sum_{m=1}^{\infty} m(m^2-1)|c_m|^2. \end{aligned}$$

On the other hand we have

$$\begin{aligned} \int_{S^1} (Hu)(u''' + u')\,d\theta &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \int_0^{2\pi} -i(\operatorname{sign} m) c_m e^{im\theta}(-in^3 + in)c_n e^{in\theta}\,d\theta \\ &= \sum_{n\in\mathbb{Z}} 2\pi i (\operatorname{sign} n) c_{-n}(-in^3 + in)c_n = \sum_{n\in\mathbb{Z}} 2\pi|n|(n^2-1)|c_n|^2 = 4\pi\sum_{n=2}^{\infty} n(n^2-1)|c_n|^2. \end{aligned}$$

□

Note that G given by (13) consists of two similar terms, and the following Theorem takes care of both at the same time as a consequence of Hilbert's double series inequality.

Theorem 8. *Suppose $f\colon S^1 \to \mathbb{R}$ is a smooth function and that $g = H(1 + \partial_\theta^2)^{-1}(f'Hf')$. Then $\|g\|_{L^\infty} \le \frac{1}{3}\|f\|^2_{\dot H^{1/2}}$.*

Proof. Expand f in a Fourier series as $f(\theta) = \sum_{n\in\mathbb{Z}} f_n e^{in\theta}$, and let $h = f'Hf'$. Then we have

$$f'Hf'(\theta) = i\sum_{m,n\in\mathbb{Z}} mn f_m f_n (\operatorname{sign} n) e^{i(m+n)\theta} = i\sum_{k\in\mathbb{Z}}\left(\sum_{n\in\mathbb{Z}} |n|\,(k-n) f_{k-n} f_n\right) e^{ik\theta} = i\sum_{k\in\mathbb{Z}} h_k e^{ik\theta},$$

where

$$h_k = \sum_{n\in\mathbb{Z}} |n|\,(k-n) f_{k-n} f_n.$$

Now let us simplify h_k: we have for $k > 0$ that

$$\begin{aligned} h_k &= \sum_{n=1}^{\infty} n(k-n) f_n f_{k-n} + \sum_{n=1}^{\infty} n(k+n)\overline{f_n} f_{k+n} \\ &= \sum_{n=1}^{k-1} n(k-n) f_n f_{k-n} + \sum_{m=1}^{\infty} (k+m)(-m) f_{k+m}\overline{f_m} + \sum_{n=1}^{\infty} n(k+n)\overline{f_n} f_{k+n}, \end{aligned}$$

where we used the substitution $m = n - k$. Clearly the middle term cancels the last term, so

$$h_k = \sum_{n=1}^{k-1} n(k-n) f_n f_{k-n}.$$

It is easy to see that $h_0 = 0$ due to cancellations, while if $k < 0$, we get

$$h_k = -\sum_{n=1}^{|k|-1} n(|k|-n)\overline{f_n f_{|k|-n}} = -\overline{h_{|k|}}.$$

Note in particular that $h_1 = h_{-1} = 0$. We thus obtain

$$f' H f'(\theta) = \sum_{k=2}^{\infty} \left(i h_k e^{ik\theta} - i\overline{h_k} e^{-ik\theta} \right),$$

so that

$$H(f' H f')(\theta) = \sum_{k=2}^{\infty} h_k e^{ik\theta} + \overline{h_k} e^{-ik\theta} = 2\mathrm{Re}\left(\sum_{k=2}^{\infty} h_k e^{ik\theta} \right).$$

It now makes sense to apply $(1+\partial_\theta^2)^{-1}$ to this function, and we obtain

$$g(\theta) = 2\mathrm{Re}\left(\sum_{k=2}^{\infty} \frac{h_k}{1-k^2} e^{ik\theta} \right),$$

so that

$$\begin{aligned}
\|g\|_{L^\infty} &\le 2\sum_{k=2}^{\infty}\sum_{n=1}^{k-1} \frac{n(k-n)|f_n||f_{k-n}|}{k^2-1} = 2\sum_{n=1}^{\infty}\sum_{k=n+1}^{\infty} \frac{n(k-n)|f_n||f_{k-n}|}{k^2-1} \\
&= 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{nm|f_n||f_m|}{(n+m)^2-1} \le \frac{4}{3}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\sqrt{nm}|f_n||f_m|}{n+m} \\
&\le \frac{4\pi}{3}\left(\sum_{n=1}^{\infty} n|f_n|^2 \right) = \frac{1}{3}\|f\|^2_{\dot{H}^{1/2}(S^1)},
\end{aligned}$$

where the inequality in the last line is precisely the well-known Hilbert double series theorem ([11], Section 9.1). □

Applying this Theorem to the terms in (13), we obtain the following straightforward Corollary which takes care of the second term in the equation (12) for u_θ in the Euler-Weil-Petersson equation.

Corollary 9. *Suppose u is a vector field on S^1, and let $G = H(1+\partial_\theta^2)^{-1}[2u_\theta H u_\theta - u_{\theta\theta} H u_{\theta\theta}]$ as in (13). Then we have*

$$\|G\|_{L^\infty} \le \frac{2}{3}\|u\|^2_{\dot{H}^{1/2}(S^1)} + \frac{1}{3}\|u\|^2_{\dot{H}^{3/2}(S^1)},$$

in terms of the degenerate seminorm $\|u\|^2_{\dot{H}^{3/2}(S^1)} = \int_{S^1} (Hu)(u''' + u')\, d\theta$.

2.3. **Proof of Theorem 3.** The work of Escher and Kolev shows that solutions of (10) are global as long as we can control the C^1 norm $\|u\|_{C^1(S^1)}$. This follows in part from the no-loss/no-gain Lemma of [7].

Lemma 10 (Escher-Kolev, "No-loss/no-gain"). *For any $s > \frac{3}{2}$, the maximal time of existence for the Euler-Weil-Petersson geodesic equation is independent of s.*

In addition the following simple computation from [8] shows that global existence for (10) in H^3 is ensured once we have a C^1 bound on u.

Proposition 11 (Escher-Kolev)**.** *If $\omega = -Hu''' - Hu'$ and ω satisfies the Euler-Arnold equation $\omega_t + u\omega_x + 2u_x\omega = 0$, then*

$$\|\omega(t)\|_{L^2} \le \|\omega_0\|_{L^2} \exp\left(-\tfrac{3}{2}\int_0^t m(s)\,ds\right),$$

where $m(s) = \inf_{x\in S^1} u_x(s,x)$.

Hence all we need to do is obtain a bound for the C^1 norm of u. Since the $\dot{H}^{3/2}$ seminorm of a solution of (10) is constant by energy conservation, it is sufficient to bound the C^1 norm in terms of the $\dot{H}^{3/2}$ seminorm. Note that the $H^{3/2}(S^1)$ norm does not in general control the $C^1(S^1)$ norm of an arbitrary function f on S^1; we need to use the special structure of the equation (10) to get this.

Proof of Theorem 3. By Proposition 5, we have

$$\big|u_\theta(t,\eta(t,\theta))\big| \le \int_0^t \big|G(s,\eta(s,\theta))\big| + \big|F(s,\eta(s,\theta))\big|\,ds \le \int_0^t 2\|u(s)\|^2_{\dot{H}^{3/2}}\,ds = 2t\|u_0\|^2_{\dot{H}^{3/2}}.$$

As a result, we know $\|u(t)\|_{L^\infty}$ remains bounded on any finite time interval, and thus we have global existence in all Sobolev spaces for $s > \frac{3}{2}$. □

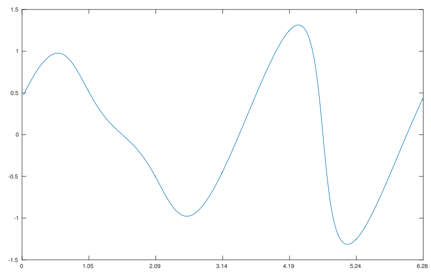
Remark 12. We proved the theorems above for solutions $u \in H^s$ for $s > \frac{3}{2}$, which by the Sobolev embedding theorem ensures that $u \in C^1$. In fact our arguments can also be extended to cover solutions with data $u_0 \in H^{3/2} \cap C^1$. On the other hand with $u_0 \in H^{3/2}$ with no assumption of continuity, we cannot ensure that the Lagrangian flow exists (see [10] for the details of the issues here), and thus our basic assumptions may fail here. Nonetheless, our techniques show that if u_0 is strictly smoother than generic $H^{3/2}$ functions, then solutions $u(t)$ remain strictly smoother; hence running time backwards, we conclude that if u_0 is a “rough” $H^{3/2}$ function which is *not* in H^s for any $s > \frac{3}{2}$, then the solution can never spontaneously become smoother than $H^{3/2}$.

3. Numerical Simulations

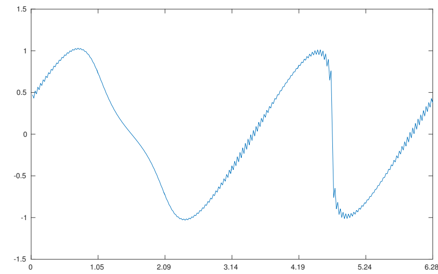
In this section we show the results of numerical simulations solving the Wunsch and Euler-Weil-Petersson equations.

3.1. **Solutions to EWP and Wunsch.** Here we implemented a Fourier-Galerkin method to get a system of ODES, coupled with a 4th order Runge-Kutta method to solve each ODE that arises. The following is a collection of solutions for the EWP and Wunsch equations with initial condition $u_0(x) = \sin(2x) + \frac{1}{2}\cos(3x)$. For each equation we have $t_0 = 0$ and $t_{\text{fin}} = .5$ (or the blowup time). (See Tables 1–4.)

Table 1. Eulerian solutions to Wunsch with $u_0 = \sin(2x) + \frac{1}{2}\cos(3x)$. Note that the slopes approach $-\infty$; after this the numerical solution appears to become singular everywhere simultaneously. It is not clear if this is what actually happens.

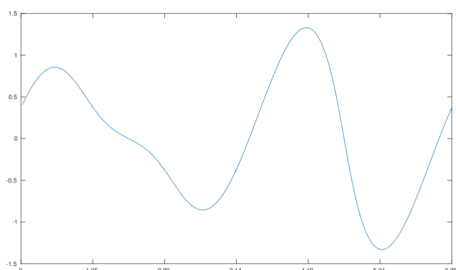


t=.125 (before blowup)

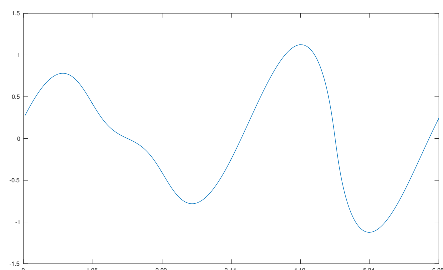


t=.25 (after blowup)

Table 2. Eulerian solutions to EWP with $u_0 = \sin(2x) + \frac{1}{2}\cos(3x)$. The profile steepens but does not become singular.

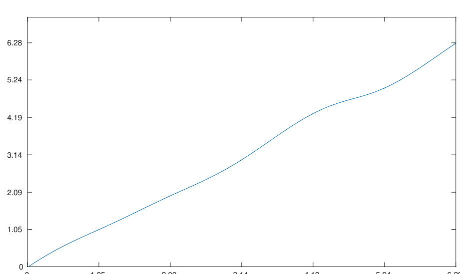


t=.25

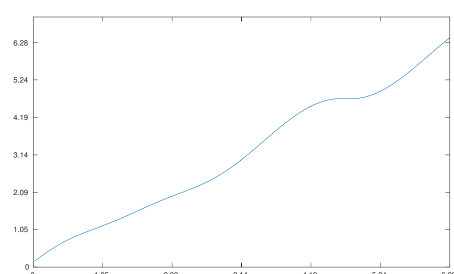


t=.5

Table 3. Lagrangian solutions to Wunsch with $u_0 = \sin(2x) + \frac{1}{2}\cos(3x)$. As u_θ approach $-\infty$, the slope of η approaches zero, and η leaves the diffeomorphism group.

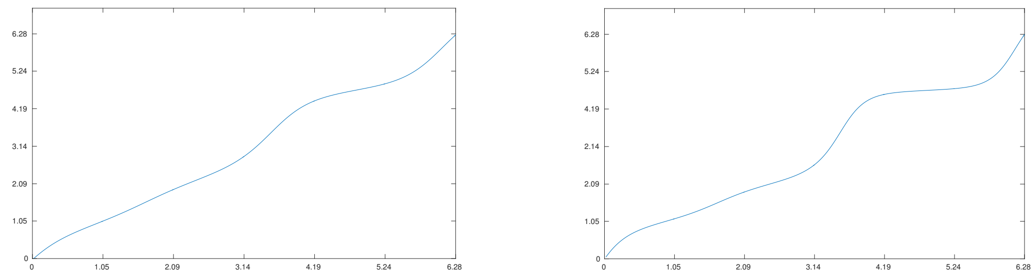


t=.125 (before blowup)



t=.25 (after blowup)

TABLE 4. Lagrangian solutions to EWP with $u_0 = \sin(2x) + \frac{1}{2}\cos(3x)$. It appears that η is flattening substantially, but the slope still remains positive.



t=.25 t=.5

References

[1] V. I. Arnold and B. Khesin. *Topological Methods in Hydrodynamics*, volume 125 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1998.

[2] M. Bauer, B. Kolev, and S. C. Preston. Geometric investigations of a vorticity model equation. *J. Differential Equations*, 260(1):478–516, 2016.

[3] A. Castro and D. Córdoba. Infinite energy solutions of the surface quasi-geostrophic equation. *Adv. Math.*, 225(4):1820–1829, 2010.

[4] A. Constantin and B. Kolev. On the geometric approach to the motion of inertial mechanical systems. *J. Phys. A*, 35(32), 2002.

[5] P. Constantin, P. D. Lax, and A. Majda. A simple one-dimensional model for the three-dimensional vorticity equation. *Comm. Pure Appl. Math.*, 38(6):715–724, 1985.

[6] D. G. Ebin and J. E. Marsden. Groups of diffeomorphisms and the motion of an incompressible fluid. *Ann. Math. (2)*, 92:102–163, 1970.

[7] J. Escher and B. Kolev. Right-invariant Sobolev metrics of fractional order on the diffeomorphism group of the circle. *J. Geom. Mech.*, 6(3):335–372, 2014.

[8] J. Escher and B. Kolev. Geodesic completeness for Sobolev H^s-metrics on the diffeomorphism group of the circle. *J. Evol. Equ.*, 14(4-5):949–968, 2014.

[9] J. Escher, B. Kolev, and M. Wunsch. The geometry of a vorticity model equation. *Commun. Pure Appl. Anal.*, 11(4):1407–1419, Jul 2012.

[10] F. Gay-Balmaz and T. Ratiu. The geometry of the universal Teichmüller space and the Euler-Weil-Petersson equation. *Adv. Math.*, 279:717–778, 2015.

[11] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities.* Cambridge University Press, London, 1934.

[12] B. Khesin and G. Misiołek. Euler equations on homogeneous spaces and Virasoro orbits. *Adv. Math.*, 176:116–144, 2003.

[13] A. A. Kirillov and D. V. Yuriev. Kähler geometry of the infinite-dimensional homogeneous space $M = \mathrm{Diff}_+(S^1)/\mathrm{Rot}(S^1)$. *Functional Anal. Appl.*, 16:114–126, 1987.

[14] J. Lenells. The Hunter-Saxton equation describes the geodesic flow on a sphere. *J. Geom. Phys.*, 57(10):2049–2064, 2007.

[15] G. Misiołek and S. C. Preston. Fredholm properties of Riemannian exponential maps on diffeomorphism groups. *Invent. Math.*, 179(1):191–227, 2010.

[16] H. Okamoto, T. Sakajo, and M. Wunsch. On a generalization of the Constantin-Lax-Majda equation. *Nonlinearity*, 21(10):2447–2461, 2008.

[17] L. Silvestre and V. Vicol. On a transport equation with nonlocal drift. *Transactions of the American Mathematical Society*, 368 (2016), no. 9, 6159-6188

[18] L. A. Takhtajan and L.-P. Teo. *Weil-Petersson Metric on the Universal Teichmüller Space*, no. 861 of *Memoirs of the American Mathematical Society.* American Mathematical Society, Providence, RI, 2006.

[19] L.-P. Teo. Velling-Kirillov metric on the universal Teichmüller curve. *J. Anal. Math.* 93(1):271–307, 2004.

[20] L.-P. Teo. Bers isomorphism on the universal Teichmüller curve. *Math. Z.* 256:603–613, 2007.

[21] A. J. Tromba. *Teichmüller Theory in Riemannian Geometry.* Springer, Basel, 1992.

[22] M. Wunsch. On the geodesic flow on the group of diffeomorphisms of the circle with a fractional Sobolev right-invariant metric. *J. Nonlinear Math. Phys.*, 17(1):7–11, 2010.

[23] S. Yamada, Local and global aspects of Weil–Petersson geometry, in *Handbook of Teichmüller Theory, Vol. IV*, 43–111, 2014.

Department of Mathematics, Brooklyn College, Brooklyn, NY 11210, USA

Department of Mathematics, CUNY Graduate Center, New York, NY 10016-4309, USA
E-mail address: `stephen.preston@brooklyn.cuny.edu`

Department of Mathematics, University of Colorado, Boulder, CO 80309-0395, USA
E-mail address: `pearce.washabaugh@colorado.edu`