CURVATURE AND CONJUGATE POINTS ON QUADRATIC LIE GROUPS

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Abstract. In this paper, we study Ricci curvature and conjugate points on Lie groups equipped with left-invariant metrics: both general Lie groups and a particular class known as quadratic Lie groups (which includes all semisimple ones). We give general results on conjugate points along geodesics in Lie groups, both in the steady and nonsteady cases. We focus particularly on two prominent families of examples coming from physics: first, $SO(n)$ with metrics that model n-dimensional rigid body dynamics for $n \geq 3$, for which we show that the Ricci curvature is diagonal and positive-definite; second, $SU(n)$ under the Zeitlin metric, which serves as a model for fluid dynamics on a sphere. Notably, we identify $SU(3)$ as an instance of a Berger-Cheeger group—Lie groups whose metrics are a deformation of a bi-invariant form along a subgroup. For Berger-Cheeger groups, we establish that their Ricci curvature admits a block decomposition and find explicit solutions to their geodesic equations.

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Contents

1. Introduction

One of the most basic questions in Riemannian geometry is the behavior of geodesics emanating from a point, and how it compares to the behavior in flat space. In particular, the presence of conjugate points, indicating the geodesic's failure to minimize length past some time, is a phenomenon linked to positive curvature.

Since the seminal work of Jacobi [34], many efforts have been made to understand the conjugate locus in various spaces and geometries, such as homogeneous Riemannian manifolds [47], symmetric, normal homogenenous and naturally reductive spaces [10, 11, 12, 55, 56], nilpotent Lie groups [52, 19, 35]; for

Lorentzian and sub-Riemannian geometries [26, 15, 4, 48]; length spaces [50]; and finally in the infinite-dimensional Riemannian setting as well [2, 3, 6, 18, 45]. In spite of all this, few criteria are known that can be used to detect conjugate points on a general Lie group with a left-invariant metric.

A much simpler framework is when one requires both left- and right-translations to be isometries. Then the Lie group admits a bi-invariant metric, and the sectional curvature is forced to be nonnegative. The bi-invariant case is therefore essentially trivial, since positive Ricci curvature yields the existence of conjugate points along any geodesic by a result of Gromoll and Meyer [25]. With only one-sided invariance, as illustrated in the well-known paper of Milnor [42], the curvature typically takes on both signs, so that standard conjugate point results that require a positive lower bound on some curvature do not apply.

An intermediate setting between the bi-invariant case and the one-sided invariant case is that of quadratic Lie groups: those admitting a non-degenerate bi-invariant (not necessarily positive definite) form. All semisimple Lie groups fall into this category, since one can take the Killing form. Two particular families of examples stand out due to their applications to physics: rigid body metrics on $SO(n)$ and the Zeitlin metric on $SU(n)$.

The metric on $SO(3)$ describing the motion of a three-dimensional free rigid body, generated by an operator containing the body's moments of inertia, goes back to Euler himself. It admits a natural generalization to n dimensions, as first pointed out by Frahm [23]. This n-dimensional generalization has since been studied from many points of view, e.g., of complete integrability in [40, 22, 46] and of stability of stationary solutions in [5, 21, 31, 32].

The Zeitlin metric on $SU(n)$ [54] gives a finite-dimensional approximation for the L^2 geometry of the volume-preserving diffeomorphism group, which describes fluid dynamics on a two-dimensional sphere [44]. In the case of $SU(3)$, the Zeitlin metric is related to a construction of Cheeger [13], where the left-invariant metric of a Lie group G is a deformation of a bi-invariant metric in the direction of a subgroup $H \subseteq G$ ($SO(3)$ in the case of the Zetilin metric). This type of deformation, which can be carried out relative to any Lie subgroup H of a Lie group G, was studied by Huizenga and Tapp in [29]. We call such groups *Berger-Cheeger*, since when $G = S^3$ and $H = S^1$ they are Berger spheres. Among the Zeitlin metrics on $SU(n)$, only the case $n = 3$ is of Berger-Cheeger type.

The aim of this paper is two-fold. First, to describe results on conjugate points on general Lie groups with a left-invariant metric, and to quadratic Lie groups in particular. Second, to study curvature, geodesics and apply the previous conjugate point results to two families of quadratic Lie groups: the family of metrics on $SO(n)$ connected to rigid bodies; and the Berger-Cheeger groups, with a special focus on Zeitlin's metric on $SU(3)$.

1.1. Main results.

Here we list the main contributions of this paper.

The first part of the paper is devoted to conjugate points. We start by considering the case of nonsteady geodesics. We prove that in a Lie group under a left-invariant metric, every closed, nonsteady geodesic develops conjugate points (Theorem 3.1). Using this theorem, we establish a link between curvature and closed geodesics, namely that if a Lie group with a left-invariant metric has dense closed geodesics, then it must have positive curvature in some section at the identity, or be abelian and flat (Corollary 3.2). This result also gives an alternative proof of the known fact that the only compact Lie group with nonpositive sectional curvature is the abelian flat torus (see Remark 3.3).

For steady geodesics, we give a necessary and sufficient criterion for conjugacy on any Lie group, provided one can solve a Sylvester equation (Theorem 3.4). This condition is fulfilled when the two operators defining the Sylvester equation do not share any eigenvalues. On a quadratic Lie group (see Section 2.2), we show that the previous criterion is particularly simple and is a consequence of a commutativity condition (Theorem 3.5). Our criterion works through a different mechanism than existing methods relying on averaging properties (cf. Gromoll & Meyer [25]) or a curvature formula (cf. Misiolek [43]), and can capture conjugate points in cases where these existing methods fail (see Remark 3.6).

The rest of the paper is devoted to two particular families of quadratic Lie groups, which have well-known applications in physics.

The first is connected to rigid bodies: on $SO(n)$, we introduce a family of metrics that are diagonal in the canonical basis of the Lie algebra of antisymmetric matrices, and that correspond to the rigid body metrics when the diagonal elements satisfy certain triangle inequalities (see Section 4). We show that the Ricci curvature of these metrics is diagonal in that same basis (Proposition 4.2), and positive in the rigid body case. If we drop the rigid body condition, then the Ricci curvature can be made negative in some directions. Nevertheless, our criterion for conjugacy along steady geodesics says that they all develop conjugate points (Proposition 4.4). It is worth noting that this result includes stable as well as unstable geodesics, such as rotations around the middle axis in three dimensions. It is not clear what happens in the nonsteady case if we drop the assumption that the metric comes from a rigid body.

The second family of quadratic Lie groups studied in this paper are Berger-Cheeger groups, a family of Lie groups where the left-invariant metric is a deformation of a bi-invariant metric in the direction of a subgroup (see Section 5). Under some additional assumptions, we show that the Ricci tensor of such groups is "block Einstein" (Proposition 5.9), in the sense that it is a constant multiple of the metric along the subalgebra, and a different multiple of the metric on its orthogonal complement. Furthermore, we show that geodesics on any Berger-Cheeger group can be computed explicitly in terms of their initial conditions and the group exponential map (Propositions 5.3 and 5.4). These formulas yield a first way to obtain conjugate points along nonsteady geodesics, starting from geodesics that are closed for the bi-invariant metric (Corollary 5.5). We then obtain a second explicit criterion for conjugacy along nonsteady geodesics (Proposition 5.7), using a method involving the index form that can be easily generalized to an arbitrary quadratic Lie group (see Appendix A). Finally, we apply this result to Berger spheres, as well as the Zeitlin model $SU(3)$ (Corollary 5.8), showing the existence of conjugate points along any nonsteady geodesic in both cases.

1.2. **Outline of the paper.** In Section 2, we provide background material on geodesics and Jacobi fields on left-invariant metric on Lie groups, with a specific focus on quadratic Lie groups. Section 3 gathers our results on conjugate points. Section 4 is devoted to generalized rigid bodies on $SO(n)$, and Section 5 to Berger-Cheeger groups in general and the Zeitlin model in particular.

2. Background

2.1. **General left-invariant metrics.** We first summarize some basic facts about general left-invariant metrics on Lie groups. We refer the reader to do Carmo [16] for further details concerning standard Riemannian geometry facts, Milnor [42] for curvature and group-theoretic properties, Arnold-Khesin [3] for stability and properties of geodesics, and Khesin et al. [36] for information about the Jacobi equation and conjugate points.

Let G be any finite-dimensional Lie group, and g an inner product on its Lie algebra $\mathfrak{g}$. It defines a left-invariant Riemannian metric on G in the following way: for any $\eta \in G$ and $x, y \in T_pG$,

$$g_\eta(x,y) := g(u,v), \quad \text{where} \quad x = \eta u,\ y = \eta v,\ u, v \in \mathfrak{g}.$$

Let $\gamma(t)$, $t \in [0,1]$, be a curve in G and $u(t) = \gamma(t)^{-1}\gamma'(t)$ its velocity vector left translated to the Lie algebra. Then γ is a geodesic for g if and only if it is a critical point of the energy functional

$$E(\gamma) := \frac{1}{2}\int_0^1 g(\gamma', \gamma')dt = \frac{1}{2}\int_0^1 g(u,u)dt,$$

where we have used the left-invariance of g. From this characterization, we can write the geodesic equations purely in terms of $u(t)$ and the operator $\mathrm{ad}^\star$ defined by the condition

$$g(\mathrm{ad}_u^\star v, w) = g(v, \mathrm{ad}_u w) \qquad \forall u, v, w \in \mathfrak{g}. \tag{2.1}$$

Proposition 2.1 ([3]). *Suppose G is a Lie group with left-invariant metric g. A curve $\gamma : [0,1] \to G$ is a geodesic for g if and only if it satisfies*

$$\gamma'(t) = \gamma(t)u(t), \qquad u'(t) = \mathrm{ad}^\star_{u(t)}u(t). \tag{2.2}$$

The following terminology, mentioned in the introduction, will be used throughout the paper, so we repeat it here in one definition.

Definition 2.2. *The first equation in* (2.2) *is called the* flow equation, *while the second is called the* Euler-Arnold *equation. When $\mathrm{ad}^\star_{u_0}u_0 = 0$, then $u(t) = u_0$ for all t, and u_0 is called a* steady *solution of the Euler-Arnold equation. If $\mathrm{ad}^\star_{u_0}u_0 \neq 0$, then $u'(t)$ is never zero, by uniqueness of solutions of ODEs, and in this case we call the solution $u(t)$* nonsteady.

Note that if u_0 is steady, then the corresponding geodesic $\gamma(t)$ is a one-parameter subgroup of G. The Euler-Arnold equation can also be rewritten as a conservation law.

Proposition 2.3 ([3]). *Suppose G is a Lie group with a left-invariant metric g. For $\eta \in G$, let $\mathrm{Ad}^\star_\eta \colon \mathfrak{g} \to \mathfrak{g}$ denote the operator defined by the condition*

$$g(\mathrm{Ad}^\star_\eta u, v) = g(u, \mathrm{Ad}_\eta v) = g(u, \eta v \eta^{-1}) \qquad \forall u, v \in \mathfrak{g}.$$

Then the Euler-Arnold equation in (2.2) may be written in the form

$$\frac{d}{dt}\Big(\mathrm{Ad}^\star_{\gamma(t)^{-1}} u(t)\Big) = 0.$$

The Riemannian exponential map $\exp\colon \mathfrak{g} \to G$ based at the identity $T_{\mathrm{id}}G \simeq \mathfrak{g}$ sends each initial condition u_0 to $\gamma(1)$, where γ solves (2.2) with $\gamma(0) = \mathrm{id}$ and $\gamma'(0) = u_0$. By homogeneity, we have $\gamma(\tau) = \exp(\tau u_0)$. Its derivative

$$D(\exp)(\tau u_0) : \mathfrak{g} \to T_{\gamma(\tau)}G \tag{2.3}$$

is always an invertible linear map for small values of τ, depending on u_0, but can become singular for large τ. A *conjugate point* along a geodesic γ is defined to be a value $\gamma(\tau) = \exp(\tau u_0)$ for which the linear map (2.3) is singular. Geometrically, this means that there exists a family of geodesics containing γ that start at $\gamma(0)$, spread out and eventually reconverge, up to first order, at $\gamma(\tau)$. Such a variation of the Riemannian exponential map defines a Jacobi field

$$J(t) = \frac{\partial}{\partial s}\Big|_{s=0} \gamma(s,t).$$

satisfying $J(0) = 0$ and $J(\tau) = 0$. A Jacobi field satisfies the Jacobi equation,

$$\frac{D^2 J}{dt^2} + R(J(t), \gamma'(t))\gamma'(t) = 0, \tag{2.4}$$

where R is the Riemann curvature tensor and $\frac{D}{dt}$ is the covariant derivative. This is obtained by linearizing the geodesic equation. On a Lie group, equation (2.4) can be translated to the Lie algebra using the group structure as follows.

Proposition 2.4. *Let G be a Lie group with left-invariant metric g, $\gamma : [0,1] \to G$ a geodesic on G, and $\gamma(s,t)$ a family of curves such that $\gamma(0,t) = \gamma(t)$ for all t. We define*

$$u(s,t) := \gamma(s,t)^{-1}\partial_t \gamma(0,t), \quad y(t) = \gamma(t)^{-1}\partial_s\gamma(0,t), \quad z(t) = \partial_s u(0,t). \tag{2.5}$$

Then $J(t) := \partial_s \gamma(0,t)$ is a Jacobi field along γ if and only if

$$y'(t) + \mathrm{ad}_{u(t)} y(t) = z(t), \qquad z'(t) = \mathrm{ad}^\star_{u(t)} z(t) + \mathrm{ad}^\star_{z(t)} u(t). \tag{2.6}$$

Thus, having a nonzero $y(t)$ as in Proposition 2.4 satisfying $y(0) = y(\tau) = 0$ is equivalent to $\gamma(0)$ and $\gamma(\tau)$ being conjugate.

For conjugate points to appear, there needs to be at least some positive sectional curvature along the geodesic. This can easily be seen by considering the index form, defined for vector fields $W(t)$ along $\gamma(t)$ that vanish at $t = 0$ and $t = \tau$ by

$$I(W,W) = \int_0^\tau g\Big(\frac{DW}{dt}, \frac{DW}{dt}\Big) - g\big(R\big(W(t), \gamma'(t)\big)\gamma'(t), W(t)\big)\, dt, \tag{2.7}$$

where R is the curvature tensor and $\frac{D}{dt}$ is the covariant derivative. A standard fact in Riemannian geometry [16] is that the index form is negative for some $W(t)$ vanishing at the endpoints if and only if there is a Jacobi field $J(t)$ which vanishes at time $t = 0$ and some at some time $t = t_0$ with $0 < t_0 < \tau$. In particular, there can be no conjugate points if the sectional curvature is everywhere non-positive, since the sectional curvature $K(W, \gamma')$ is proportional to $g(R(W, \gamma')\gamma', W)$. On a Lie group, this quantity can be computed by Arnold's formula [3]

$$g(R(u,v)v,u) = \frac{1}{4}\|\mathrm{ad}^\star_u v + \mathrm{ad}^\star_v u\|^2 - g(\mathrm{ad}^\star_u u, \mathrm{ad}^\star_v v) - \frac{3}{4}\|\mathrm{ad}_u v\|^2 + \frac{1}{2} g(\mathrm{ad}_u v, \mathrm{ad}^\star_v u - \mathrm{ad}^\star_u v),$$

for any $u, v \in \mathfrak{g}$. In the sequel, it will be helpful to use a slightly different formula, as in [38],

$$g(R(u,v)v,u) = \frac{1}{4}\|\mathrm{ad}^\star_u v + \mathrm{ad}^\star_v u + \mathrm{ad}_u v\|^2 - g(\mathrm{ad}^\star_u v + \mathrm{ad}_u v, \mathrm{ad}_u v) - g(\mathrm{ad}^\star_u u, \mathrm{ad}^\star_v v). \tag{2.8}$$

As for the index form (2.7), it can be written in the following way.

Proposition 2.5. *In the notations of Proposition 2.4, the index form is given by*

$$I(y,y) = \int_0^\tau g(z(t),z(t)) - g(\mathrm{ad}_{y(t)}z(t),u(t))\,dt, \tag{2.9}$$

for variations $y(t)$ which vanish at $t=0$ and $t=\tau$. If there is such a y and $\tau>0$ such that $I(y,y)<0$, then there is a conjugate point occurring at a time $t=t_0<\tau$.

A well-known method to find conjugate points that relies on the index form is the Misiołek criterion, particularly effective for finding conjugate points along steady flows of the Euler equation for ideal fluids [18, 43, 45], which can also be used for some nonsteady flows such as Rossby-Haurwitz waves [6]. Although it has so far only been applied in the context of geometric hydrodynamics, it can be readily generalized to any Lie group as follows.

Given an initial condition u_0 which determines a steady solution of the Euler-Arnold equation, let $y(t) = f(t)v$ be a variation vector field, for some fixed vector v and scalar function $f(t)$. Then $z(t) = f'(t)v + f(t)\mathrm{ad}_{u_0}v$, and the index form (2.9) becomes

$$\begin{aligned} I(y,y) &= \int_0^\tau (f')^2\|v\|^2 + 2ff'g\big(v,\mathrm{ad}_{u_0}v\big) + f^2\|\mathrm{ad}_{u_0}v\|^2 - f^2 g\big(\mathrm{ad}_v(\mathrm{ad}_{u_0}v),u_0\big)\,dt \\ &= \int_0^\tau (f')^2\|v\|^2 + 2ff'g\big(v,\mathrm{ad}_{u_0}v\big) + f^2 g\big(\mathrm{ad}_v u_0 + \mathrm{ad}_v^* u_0, \mathrm{ad}_v u_0\big)\,dt. \end{aligned}$$

Now choosing $f(t) = \sin\left(\frac{\pi t}{\tau}\right)$ and taking $\tau>0$ large, the first term can be made arbitrarily small since $f' \sim 1/\tau$ and the second even integrates to zero, so if

$$g\big(\mathrm{ad}_v u_0 + \mathrm{ad}_v^* u_0, \mathrm{ad}_v u_0\big) < 0 \qquad \text{for some } v\in\mathfrak{g}, \tag{2.10}$$

then the geodesic with initial velocity u_0 will eventually develop conjugate points. Condition (2.10) is precisely the Misiołek criterion. Comparing it with formula (2.8), and using the fact that u_0 is a steady solution, we see that (2.10) directly implies positive curvature on the 2-plane spanned by u_0 and v.

In 4.2, we give new examples on generalized rigid bodies where the Misiołek criterion does not apply even in the steady case, but Theorem 3.5 is able to detect conjugate points.

2.2. **Quadratic Lie groups.** A special focus of this paper is on quadratic Lie groups. A Lie group G is said to be *quadratic* if there is a nondegenerate bilinear form $\langle\cdot,\cdot\rangle$ (not necessarily positive-definite) on the Lie algebra, which has the property of being bi-invariant

$$\langle \mathrm{ad}_u v, w\rangle + \langle v, \mathrm{ad}_u w\rangle = 0 \qquad \text{for all } u,v,w\in\mathfrak{g}.$$

Not every Lie group has such a bilinear form: compact and abelian groups have positive-definite ones [42], and more generally any semisimple Lie group has a nondegenerate Killing form which is bi-invariant [27], while for example the upper half-plane considered as a Lie group has none.

Every left-invariant Riemannian metric on a quadratic group G can be defined from an invertible symmetric operator Λ on $\mathfrak{g}$

$$g(u,v) = \langle u, \Lambda v\rangle. \tag{2.11}$$

In this case, the operator $\mathrm{ad}^\star$ can be written in terms of Λ.

Lemma 2.6. *If $\langle\cdot,\cdot\rangle$ is a nondegenerate bi-invariant bilinear form on $\mathfrak{g}$, and if Λ is a symmetric operator such that $g(u,v) := \langle u,\Lambda v\rangle$ defines a Riemannian metric, then*

$$\mathrm{ad}_u^\star v = -\Lambda^{-1}(\mathrm{ad}_u \Lambda v). \tag{2.12}$$

We can now express the Euler-Arnold equation in the quadratic case.

Lemma 2.7. *If $\langle\cdot,\cdot\rangle$ is a bi-invariant bilinear form on $\mathfrak{g}$, and if Λ is a symmetric operator such that $g(u,v) := \langle u,\Lambda v\rangle$ defines a Riemannian metric, then the geodesic equation is given by*

$$\gamma'(t) = \gamma(t)u(t), \qquad \Lambda u'(t) + \mathrm{ad}_{u(t)}\Lambda u(t) = 0. \tag{2.13}$$

Similarly, the Jacobi equation and the index form can be written in terms of the Λ operator.

Lemma 2.8. *Let G be a quadratic Lie group with left-invariant metric induced by an operator Λ. Then in the notations of Proposition 2.4, the Jacobi equation* (2.6) *is written*

$$y'(t) + \mathrm{ad}_{u(t)}y(t) = z(t), \qquad \Lambda z'(t) + \mathrm{ad}_{u(t)}\Lambda z(t) + \mathrm{ad}_{z(t)}\Lambda u(t) = 0. \tag{2.14}$$

The index form is given by

$$I(y,y) = \int_0^\tau \langle \Lambda z(t) + \mathrm{ad}_{y(t)}\Lambda u(t), z(t)\rangle\, dt, \tag{2.15}$$

for variations $y(t)$ which vanish at $t = 0$ and $t = \tau$.

3. Conjugate points on general Lie groups

3.1. **Nonsteady geodesics.** We first prove the existence of conjugate points along any closed, nonsteady geodesic, then discuss a simple consequence: a compact nonabelian Lie group with a left-invariant metric must have some positive sectional curvature. Of course every closed geodesic has a *cut* point, since the geodesic stops minimizing, but a conjugate point requires *nearby* geodesics that are shorter.

Theorem 3.1. *Suppose $\gamma(t)$ is a solution of* (2.2)*, with velocity $u(t)$ nonconstant. If right multiplication by $\gamma(\tau)\gamma(0)^{-1}$ is an isometry of the left-invariant metric for some $\tau > 0$, then $\gamma(\tau)$ is conjugate to $\gamma(0)$. In particular this applies if $\gamma(\tau) = \gamma(0)$.*

Proof of Theorem 3.1. By left-invariance, we may assume without loss of generality that $\gamma(0)$ is the identity. Let $u(t)$ be defined by the flow equation $\gamma'(t) = \gamma(t)u(t)$. By Proposition 2.3, $u(t)$ satisfies the angular momentum conservation law

$$u(t) = \mathrm{Ad}^\star_{\gamma(t)}u_0 \tag{3.1}$$

in terms of the initial condition $u(0) = u_0$. Differentiating the Euler equation (2.1) for $u(t)$ with respect to time gives

$$u''(t) - \mathrm{ad}^\star_{u(t)}u'(t) - \mathrm{ad}^\star_{u'(t)}u(t) = 0,$$

showing that $z_p(t) := u'(t)$ is a particular solution of the linearized Euler equation (2.6). Since $u'(t)$ is nowhere zero, this is a nontrivial solution. If we rewrite (2.6) in the form

$$\frac{d}{dt}\Big(\mathrm{Ad}_{\gamma(t)}y(t)\Big) = \mathrm{Ad}_{\gamma(t)}z(t),$$

we see that the general solution of the complementary homogeneous equation is $y_c(t) = \mathrm{Ad}_{\gamma(t)^{-1}}w_0$ for some vector $w_0 \in \mathfrak{g}$. Thus the general solution is

$$y(t) = y_p(t) + y_c(t) = u(t) + \mathrm{Ad}_{\gamma(t)^{-1}}w_0.$$

To have $y(0) = 0$ as desired, we choose $w_0 = -u_0$. Inserting (3.1) in this, we obtain

$$y(t) = \mathrm{Ad}^\star_{\gamma(t)}u_0 - \mathrm{Ad}_{\gamma(t)^{-1}}u_0.$$

If right multiplication by $\gamma(\tau)$ is an isometry, then $\mathrm{Ad}^\star_{\gamma(\tau)}\mathrm{Ad}_{\gamma(\tau)} = I$, and we will obtain $y(\tau) = 0$. Clearly the corresponding Jacobi field $y(t)$ is nontrivial on $[0,\tau]$ since $z_p(t)$ is nontrivial. □

In order to state a Corollary of Theorem 3.1, recall that a Riemannian manifold M is said to have *dense closed geodesics* if for every $p \in M$ and every unit $v \in T_pM$ and every $\varepsilon > 0$, there is a $w \in T_pM$ such that $|v - w| < \varepsilon$ and the geodesic $\gamma(t) = \exp_p(tw)$ is closed (i.e., $\gamma(\tau) = p$ for some $\tau > 0$).

Examples of manifolds with dense closed geodesics include: compact manifolds with negative curvature, by the Anosov theorem; certain quotients of nilpotent Lie groups, all of which have curvatures of both signs (so long as they are not abelian) ([14], [19], [39], [41]); and of course many examples with positive curvature, such as $U(n)$ with the bi-invariant metric (see also [8]). By left-invariance, it is enough to check this condition when p is the identity.

Corollary 3.2. *If a (finite-dimensional) Lie group G with left-invariant metric has dense closed geodesics, then either it has positive curvature in some section at the identity, or it is abelian and flat.*

Proof. For every $v \in T_{\rm id}G$ there is a nearby vector w such that the geodesic in the direction of w is closed. If any such geodesic is nonsteady, the previous theorem gives a conjugate point along it, which implies there must be positive curvature somewhere along the geodesic. Otherwise every closed geodesic is steady.

If the geodesic $t \mapsto \exp_{\rm id}(tw)$ is steady, then we must have $\mathrm{ad}^\star_w w = 0$. Now the quadratic form $v \mapsto A(v) := \mathrm{ad}^\star_v v$ is continuous on a finite-dimensional Lie algebra, and density of closed geodesics implies that for every $v \in T_{\rm id}G$ and $\delta > 0$ there is a $w \in T_{\rm id}G$ such that $|v - w| < \delta$ and $A(w) = 0$. Hence we must have $A(v) = 0$ for all $v \in T_{\rm id}G$.

Now $\mathrm{ad}^\star_v v = 0$ for all $v \in T_{\rm id}G$ implies that the metric is bi-invariant, and thus the sectional curvature $K(u, v)$ is given by the well-known formula $K(u, v) = \frac{1}{4}|[u, v]|^2$. Thus either $[u, v] = 0$ for all $u, v \in T_{\rm id}G$, so that G is abelian and flat, or we again get some positive curvature. □

Remark 3.3. *This result also gives an alternative proof of the known fact that the only compact Lie group with nonpositive sectional curvature is the abelian flat torus. This arises from the fact that nonpositive curvature requires solvability of the group, and the only compact solvable groups are abelian – see, e.g.,* [37].

3.2. **Steady geodesics.** Here we prove our criterion for conjugate points along steady geodesics for general left-invariant metrics.

Theorem 3.4. *Suppose u_0 is a steady solution of the Euler-Arnold equation (2.2), i.e., that $\mathrm{ad}^\star_{u_0} u_0 = 0$. Let L and F be the linear operators on the Lie algebra $\mathfrak{g}$ defined by*

$$L(v) := \mathrm{ad}_{u_0} v, \qquad F(v) = \mathrm{ad}^\star_{u_0} v + \mathrm{ad}^\star_v u_0.$$

Then both operators map the g-orthogonal complement of u_0 to itself. If there is an operator R defined on this orthogonal complement such that $RF + LR = I$, then there is a conjugate point along the geodesic if and only if for some $\tau > 0$ we have

$$\det\left(e^{\tau L} R e^{\tau F} - e^{-\tau L} R e^{-\tau F}\right) = 0.$$

Proof. Let $u_0 \in \mathfrak{g}$ be a steady solution of the Euler-Arnold equation (2.2), i.e. such that $\mathrm{ad}^\star_{u_0} u_0 = 0$. There are conjugate points along the corresponding geodesic γ if there is a time $\tau > 0$ and vector fields $y(t)$ and $z(t)$ of the Lie algebra satisfying $y(0) = y(\tau) = 0$ and the system of equations (2.5), that is,

$$y' + L(y) = z, \qquad z' = F(z) \tag{3.2}$$

in terms of the linear operators on the Lie algebra $\mathfrak{g}$

$$L(y) = \mathrm{ad}_{u_0} y, \qquad F(y) = \mathrm{ad}^\star_{u_0} y + \mathrm{ad}^\star_y u_0. \tag{3.3}$$

Both these operators map into the g-orthogonal complement of u_0. Indeed, for any $y \in \mathfrak{g}$,

$$g(L(y), u_0) = g(y, \mathrm{ad}^\star_{u_0} u_0) = 0$$
$$g(F(y), u_0) = g(y, \mathrm{ad}_{u_0} u_0) + g(u_0, \mathrm{ad}_y u_0) = -g(\mathrm{ad}^\star_{u_0} u_0, y) = 0.$$

Now, if there is an operator R defined on this orthogonal complement such that $RF + LR = I$, then every solution of (3.2) with $y(0), z(0) \in u_0^\perp$ is obtained as

$$y(t) = R e^{tF} x_0 + e^{-tL} w_0,$$

for some choice of $w_0, x_0 \in u_0^\perp$, since the first part solves the nonhomogeneous equation for y, while the second part solves the homogeneous equation.

A conjugate point eventually occurs if and only if there is a time τ such that $y(0) = y(2\tau) = 0$, or equivalently, $y(-\tau) = y(\tau)$, since the geodesic is steady and this time translation has no impact. This means

$$R e^{\tau F} x_0 + e^{-\tau L} w_0 = 0$$
$$R e^{-\tau F} x_0 + e^{\tau L} w_0 = 0$$

Left multiplying by $e^{\tau L}$ and $e^{-\tau L}$ respectively the first and second equation, and eliminating w_0, we obtain that the condition for the existence of a conjugate point along the steady geodesic γ is the existence of a nontrivial $x_0 \in \mathfrak{g}$ such that

$$(e^{\tau L} R e^{\tau F} - e^{-\tau L} R e^{-\tau F}) x_0 = 0,$$

which is equivalent to asking that the determinant of the matrix multiplying x_0 be zero, as claimed. □

In practice we will see that this form is especially convenient when L and F are block-diagonal matrices, so that the matrix exponentials can be computed easily.

3.3. **Steady geodesics in quadratic Lie groups.** In this section, we specialize to quadratic Lie groups, and give a criterion for conjugate points along the simplest steady geodesics: those generated by eigenvectors of the operator Λ. For a conjugacy criterion along nonsteady geodesics in a quadratic Lie group, see Appendix A. For simplicity we will assume the underlying bi-invariant form is positive-definite to make the constants appearing in the proof positive and simplify the formulas, though this assumption is not necessary.

Theorem 3.5. *Suppose G is a quadratic group, with a left-invariant metric g given by* (2.11) *for an operator Λ with a bi-invariant metric $\langle\cdot,\cdot\rangle$. Let u_0 be an eigenvector of Λ with $\Lambda u_0 = \lambda u_0$, and let $L = \mathrm{ad}_{u_0}$. If L^2 commutes with Λ, then there is eventually a conjugate point along the geodesic $\gamma(t)$ solving the geodesic equation* (2.13).

Proof. By assumption, both L^2 and Λ can be diagonalized in the same basis $\{w_1,\dots,w_n\}$, orthonormal with respect to the bi-invariant metric. Since L is antisymmetric in the bi-invariant metric, we know L^2 is symmetric and nonpositive. Thus, there exists a basis $\{w_1,\dots,w_{2m},w_{2m+1},\dots,w_n\}$ of eigenvectors of L such that

$$L(w_{2j-1}) = \epsilon_j w_{2j}, \qquad L(w_{2j}) = -\epsilon_j w_{2j-1},$$
$$\Lambda(w_{2j-1}) = \alpha_j w_{2j-1}, \qquad \Lambda(w_{2j}) = \beta_j w_{2j}, \qquad 1\le j\le m,$$

and

$$L(w_k) = 0 \qquad \Lambda(w_k) = \gamma_k w_k, \qquad 2m+1\le k\le n,$$

with the vector u_0 itself a constant multiple of one of the w_k, and all numbers α, β, γ, and ϵ are strictly positive. See for example Greub [24], Section 8.4.

We now compute the operators defined by (3.3). Here we have $L(z) = \mathrm{ad}_{u_0} z$ and

$$\begin{aligned} F(z) &= \mathrm{ad}^\star_{u_0}(z) + \mathrm{ad}^\star_z u_0 = -\Lambda^{-1}(\mathrm{ad}_{u_0}\Lambda z + \mathrm{ad}_z \Lambda u_0)\\ &= -\Lambda^{-1}(\mathrm{ad}_{u_0}\Lambda z + \lambda \mathrm{ad}_z u_0)\\ &= -\Lambda^{-1} L(\Lambda - \lambda I)(z), \end{aligned}$$

where we have used (2.12). F is given in the orthonormal basis $\{w_1,\dots,w_n\}$ by

$$F(w_{2j-1}) = -\frac{\epsilon_j(\alpha_j-\lambda)}{\beta_j} w_{2j}, \qquad F(w_{2j}) = \frac{\epsilon_j(\beta_j-\lambda)}{\alpha_j} w_{2j-1}, \qquad 1\le j\le m,$$
$$F(w_k) = 0, \qquad 2m+1\le k\le n.$$

Now let X denote the span of $\{w_1,\dots,w_{2m}\}$, and observe that L, Λ, and F all map X to itself. We thus restrict F and L to X and note that L is invertible on this subspace, so that

$$R = \lambda^{-1}L^{-1}\Lambda \quad\Longrightarrow\quad RF + LR = I.$$

In the basis, R is given by

$$R(w_{2j-1}) = -\frac{\alpha_j}{\lambda\epsilon_j} w_{2j}, \qquad R(w_{2j}) = \frac{\beta_j}{\lambda\epsilon_j} w_{2j-1}.$$

We thus find that the matrices L, R, and F restricted to X can be written as $(2m)\times(2m)$ matrices consisting of 2×2 nonzero blocks along the diagonal, and zero everywhere else. Hence the matrices e^{tL} and e^{tF} also split this way, and the determinant from Theorem 3.4 splits into a product:

$$\det\left(e^{tL}Re^{tF} - e^{-tL}Re^{-tF}\right) = \prod_{j=1}^{k} \det\left(e^{tL_j}R_j e^{tF_j} - e^{-tL_j}R_j e^{-tF_j}\right),$$

where each 2×2 matrix acting in $X_j = \mathrm{span}\{w_{2j-1}, w_{2j}\}$ is given by

$$L_j = \begin{pmatrix} 0 & -\epsilon_j \\ \epsilon_j & 0\end{pmatrix}, \qquad F_j = \begin{pmatrix} 0 & \frac{\epsilon_j(\beta_j-\lambda)}{\alpha_j} \\ -\frac{\epsilon_j(\alpha_j-\lambda)}{\beta_j} & 0\end{pmatrix}, \qquad R_j = \begin{pmatrix} 0 & \frac{\beta_j}{\lambda\epsilon_j} \\ -\frac{\alpha_j}{\lambda\epsilon_j} & 0\end{pmatrix}.$$

Obviously the product of determinants is zero if and only if at least one of them is zero; thus it is sufficient to fix a $j\in\{1,\dots,m\}$. The matrix exponential of tL_j is easy: since L_j is always antisymmetric, we have

$$e^{tL_j} = \cos(\epsilon_j t) I + \epsilon_j^{-1}\sin(\epsilon_j t) L_j.$$

On the other hand the form of the matrix exponential of τF_j depends on the quantity

$$d_j = \det F_j = \frac{\epsilon_j^2(\beta_j - \lambda)(\alpha_j - \lambda)}{\alpha_j \beta_j}.$$

We have $e^{tF_j} = c_j(t)I + s_j(t)F$, where the pair of generalized trigonometric functions are

$$\big(c_j(t), s_j(t)\big) := \begin{cases} (\cosh rt, r^{-1}\sinh rt) & \text{if } d_j = -r^2, \\ (1, t) & \text{if } d_j = 0, \\ (\cos rt, r^{-1}\sin rt) & \text{if } d_j = r^2. \end{cases}$$

It is then easy to see that each matrix $e^{tL_j}R_j e^{tF_j} - e^{-tL_j}R_j e^{-tF_j}$ is *diagonal*, and so the condition in Theorem 3.4 becomes simply verifying that at least one of the following quantities is zero at some $t = \tau$:

$$f_j(t) = \sin(\epsilon_j t)c_j(t) - \frac{\epsilon_j(\alpha_j - \lambda)}{\alpha_j}\, s_j(t)\cos(\epsilon_j t)$$
$$g_j(t) = \sin(\epsilon_j t)c_j(t) - \frac{\epsilon_j(\beta_j - \lambda)}{\beta_j}\, s_j(t)\cos(\epsilon_j t).$$

Note that for small positive values of t, both these functions are small and positive.

If $d_j = r^2 > 0$, then $(\alpha_j - \lambda)$ and $(\beta_j - \lambda)$ have the same sign. We assume without loss of generality that they are both positive. The functions become

$$f_j(t) = \sin(\epsilon_j t)\cos(rt) - \sqrt{\frac{\beta_j(\alpha_j - \lambda)}{\alpha_j(\beta_j - \lambda)}}\,\sin(rt)\cos(\epsilon_j t)$$
$$g_j(t) = \sin(\epsilon_j t)\cos(rt) - \sqrt{\frac{\alpha_j(\beta_j - \lambda)}{\beta_j(\alpha_j - \lambda)}}\,\sin(rt)\cos(\epsilon_j t).$$

We can simplify the following positive linear combination of $f(t)$ and $g(t)$ to get

$$\sqrt{\alpha_j(\beta_j - \lambda)}\, f_j(t) + \sqrt{\beta_j(\alpha_j - \lambda)}\, g_j(t) = \left(\sqrt{\alpha_j(\beta_j - \lambda)} + \sqrt{\beta_j(\alpha_j - \lambda)}\right)\sin(\epsilon_j - r)t.$$

Since we can obviously make this negative for some choice $t = \tau$, we find that at least one of $f_j(t)$ or $g_j(t)$ must have been negative at this time. Hence at least one of them must have crossed the axis already.

The situation when $d_j = -r^2$ is simpler, when $(\alpha_j - \lambda)$ and $(\beta_j - \lambda)$ have opposite signs. Again assuming without loss of generality that $\alpha_j - \lambda > 0$, the function f_j is

$$f_j(t) = \sin(\epsilon_j t)\cosh(rt) - \sqrt{\frac{\beta_j(\alpha_j - \lambda)}{\alpha_j(\lambda - \beta_j)}}\,\sinh(rt)\cos(\epsilon_j t),$$

and we observe that

$$f_j\left(\frac{2\pi}{\epsilon_j}\right) = -\sqrt{\frac{\beta_j(\alpha_j - \lambda)}{\alpha_j(\lambda - \beta_j)}}\,\sinh\left(\frac{2\pi r}{\epsilon_j}\right) < 0,$$

so that $f_j(t)$ must have crossed the axis before τ. The cases where $\alpha_j < \lambda$ or $\alpha_j = \lambda$ are straightforward using the same approach. □

In the context of Theorem 3.5 above, we make the following remark on the Misiołek criterion (2.10), which explains why it captures some conjugate points but not others, and its connection with stability. Note that we have stability of u_0 in the Eulerian sense if and only if all eigenvalues of F are nonpositive, corresponding to $d_j \le 0$ for all j. This happens if for example λ is smaller than all α_j, β_j or larger than all of them.

Remark 3.6. *If u_0 is a steady solution of the Euler equation on a quadratic Lie group, corresponding to eigenvalue λ of Λ, the Misiołek criterion is written in terms of $L = \mathrm{ad}_{u_0}$ as*

$$0 > g(\mathrm{ad}_v u_0 + \mathrm{ad}_v^\star u_0, \mathrm{ad}_v u_0) = \langle \Lambda \mathrm{ad}_v u_0 - \lambda \mathrm{ad}_v u_0, \mathrm{ad}_v u_0\rangle = \langle (\Lambda - \lambda I)Lv, Lv\rangle.$$

Thus, we see that for steady solutions corresponding to, e.g., the smallest eigenvalue λ of Λ, the Misiołek criterion fails to detect the existence of conjugate points, that do exist when L^2 commutes with Λ, as for the generalized rigid body metric studied in the following section.

4. Generalized rigid body metric on rotations

The motion of a three-dimensional free rigid body has a long history, going back to Euler himself. Ignoring translations, at any point in time the body is rotating around an axis, but the axis itself can vary with time, which leads to fairly complicated phenomena. Already in three dimensions, the general equations of motion cannot be integrated by elementary functions, but require the so-called Jacobi elliptic functions [33].

In this section, we begin by showing that the Ricci curvature of the kinetic energy metric describing the n-dimensional rigid body system on $SO(n)$ is everywhere positive, generalizing a result of Milnor [42] when $n = 3$, and leading to the existence of conjugate points along any geodesic, steady or nonsteady, by a result of Gromoll and Meyer [25]. This motivates the study of a larger class of metrics on $SO(n)$ that generalize the classical rigid body metrics in a natural way, while admitting some negative Ricci curvatures. We will see that our criterion finds conjugate points along steady geodesics even in the presence of negative curvature. The main takeaway here is that our criterion works through a different mechanism than existing methods, since it detects conjugate points not by an averaging method (cf. Gromoll & Meyer [25]), and is also not tied to a specific curvature formula or spectral condition (see Remark 3.6).

Consider the group of rotations $SO(n)$. A basis of its Lie algebra $\mathfrak{so}(n)$ is given by the matrices $\{e_{ij}\}_{1\le i<j\le n}$, where e_{ij} is the matrix full of zeros except for -1 in position (i, j) and 1 in position (j, i). We consider a symmetric operator $\Lambda : \mathfrak{so}(n) \to \mathfrak{so}(n)$ with eigenvectors e_{ij}, i.e., such that there exist real numbers λ_{ij} with

$$\Lambda e_{ij} = \lambda_{ij} e_{ij}, \qquad 1 \le i < j \le n.$$

For convenience in the curvature formula later, we define $\lambda_{ji} = \lambda_{ij}$. We equip $SO(n)$ with the left-invariant Riemannian metric generated on the Lie algebra by

$$g(u, v) := \langle u, \Lambda v\rangle, \quad \text{where} \quad \langle u, v\rangle = \tfrac{1}{2}\mathrm{Tr}(uv^\top), \qquad u, v \in \mathfrak{so}(n), \tag{4.1}$$

is the bi-invariant metric and Tr is the trace. We refer to this metric as the generalized rigid body metric, since when $\Lambda(u) := \frac{1}{2}(Mu + uM)$ with M a symmetric matrix with positive eigenvalues $\mu_1, \dots, \mu_n$, it is the kinetic energy metric, describing the rotations of a rigid body with principal moments of inertia μ_i. We will consider this special case at the end of this section.

4.1. Ricci curvature.

We start by computing the Ricci curvature. We first need the following lemma.

Lemma 4.1. *The Lie brackets between the eigenvectors $\{e_{ij}\}_{1\le i<j\le n}$ are given by*

$$\begin{aligned} &\forall\, i<j<k, \quad [e_{ij}, e_{ik}] = e_{jk}, \quad [e_{ik}, e_{jk}] = e_{ij}, \quad [e_{jk}, e_{ij}] = e_{ik}, \\ &\textit{if} \quad \{i, j\} \cap \{k, \ell\} = \emptyset, \quad \textit{then} \quad [e_{ij}, e_{k\ell}] = 0. \end{aligned} \tag{4.2}$$

Proof. For any i, j, k, ℓ and any p, q, we have that the (p, q) component of e_{ij} is given by $(e_{ij})_{pq} = -\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}$, and a straightforward computation gives

$$[e_{ij}, e_{k\ell}]_{pq} = \sum_m (e_{ij})_{pm}(e_{k\ell})_{mq} - \sum_n (e_{k\ell})_{pn}(e_{ij})_{nq} = (\delta_{ik}e_{j\ell} - \delta_{jk}e_{i\ell} + \delta_{j\ell}e_{ik} + \delta_{i\ell}e_{kj})_{pq}.$$

□

Now we can compute the Ricci curvature.

Proposition 4.2. *The Ricci curvature tensor of the generalized rigid body metric* (4.1) *on $SO(n)$ is diagonal in the basis $\{e_{ij}\}_{1\le i<j\le n}$ with diagonal terms*

$$\mathrm{Ric}(e_{ij}, e_{ij}) = \sum_{k\ne i,j} \frac{(\lambda_{ij} - \lambda_{ik} + \lambda_{jk})(\lambda_{ij} + \lambda_{ik} - \lambda_{jk})}{2\lambda_{ik}\lambda_{jk}}$$

Remark 4.3. *In particular, for the standard case where the metric corresponds to the kinetic energy of a rigid body with moments of inertia $\mu_1, \dots \mu_n > 0$, then $\lambda_{ij} = \frac{\mu_i+\mu_j}{2}$ and we obtain*

$$\mathrm{Ric}(e_{ij}, e_{ij}) = \sum_{k\ne i,j} \frac{2\mu_i\mu_j}{(\mu_i + \mu_k)(\mu_j + \mu_k)}.$$

Thus the Ricci curvature of the rigid body metric on $SO(n)$ is everywhere positive.

Proof. Using Lemmas 2.6 and 4.1, we see that the operator $\mathrm{ad}^\star$ takes the following values:

$$\begin{aligned} \forall\, i<j<k, \quad \mathrm{ad}^\star_{e_{ik}} e_{ij} &= \frac{\lambda_{ij}}{\lambda_{jk}} e_{jk}, \quad \mathrm{ad}^\star_{e_{jk}} e_{ik} = \frac{\lambda_{ik}}{\lambda_{ij}} e_{ij}, \quad \mathrm{ad}^\star_{e_{ij}} e_{jk} = \frac{\lambda_{jk}}{\lambda_{ik}} e_{ik}, \\ \mathrm{ad}^\star_{e_{ij}} e_{ik} &= -\frac{\lambda_{ik}}{\lambda_{jk}} e_{jk} \quad \mathrm{ad}^\star_{e_{ik}} e_{jk} = -\frac{\lambda_{jk}}{\lambda_{ij}} e_{ij} \quad \mathrm{ad}^\star_{e_{jk}} e_{ij} = -\frac{\lambda_{ij}}{\lambda_{ik}} e_{ik} \end{aligned} \tag{4.3}$$

and all others are zero. Our goal is to compute the Ricci curvature in this orthogonal basis, and we recall that for any vectors v and w, we have

$$\mathrm{Ric}(v,w) = \sum_{k<\ell} \frac{g\big(R(e_{k\ell},v)w, e_{k\ell}\big)}{\lambda_{k\ell}}.$$

Polarizing formula (2.8) for the curvature, fixing u, we have

$$\begin{aligned} g(R(u,v)w,u) = \frac{1}{4} g(\mathrm{ad}^\star_u v + \mathrm{ad}^\star_v u + \mathrm{ad}_u v, \mathrm{ad}^\star_u w + \mathrm{ad}^\star_w u + \mathrm{ad}_u w) - \tfrac{1}{2} g(\mathrm{ad}^\star_u v + \mathrm{ad}_u v, \mathrm{ad}_u w) \\ - \tfrac{1}{2} g(\mathrm{ad}^\star_u w + \mathrm{ad}_u w, \mathrm{ad}_u v) - \tfrac{1}{2} g(\mathrm{ad}^\star_u u, \mathrm{ad}^\star_v w + \mathrm{ad}^\star_w v). \end{aligned} \tag{4.4}$$

Thus when we consider $u = e_{k\ell}$, the last term will be zero since $\mathrm{ad}^\star_u u = 0$. Furthermore if $v = e_{ij}$, then by formulas (4.2) and (4.3), the only way to have $\mathrm{ad}^\star_u v$, $\mathrm{ad}^\star_v u$, or $\mathrm{ad}_u v$ nonzero is if $\{i,j\} \cap \{k,\ell\}$ has exactly one element, and all of those terms are proportional to the same basis element. Thus the only way to have either of these inner products nonzero is if w is the same basis element as v. Since this is true for every $u = e_{k\ell}$, we conclude that the Ricci curvature is diagonal in the basis $\{e_{ij}\}$.

When performing the sum with fixed $v = e_{ij}$ for $i<j$, it is therefore sufficient to compute only terms of the form $g(R(u,v)v,u)$ when either $k<i<j$ with $u = e_{ki}$ or $u = e_{kj}$; or $i<k<j$ with $u = e_{ik}$ or $u = e_{kj}$; or $i<j<k$ with $u = e_{ik}$ or $u = e_{jk}$. The computations are similar in the three ranges for k, and we end up with the same answer, so we will only present this last case.

The Ricci curvature is given in the orthogonal basis by

$$\mathrm{Ric}(e_{ij}, e_{ij}) = \sum_{k\neq i,j} \frac{1}{\lambda_{ik}} g\big(R(e_{ij}, e_{ik})e_{ik}, e_{ij}\big) + \sum_{k\neq i,j} \frac{1}{\lambda_{jk}} g\big(R(e_{ij}, e_{jk})e_{jk}, e_{ij}\big). \tag{4.5}$$

Fixing $k>j$ and using (2.8), we get

$$\begin{aligned} g\big(R(e_{ij}, e_{ik})e_{ik}, e_{ij}\big) &= \tfrac{1}{4}\big\|\mathrm{ad}^\star_{e_{ij}} e_{ik} + \mathrm{ad}^\star_{e_{ik}} e_{ij} + [e_{ij}, e_{ik}]\big\|^2 - g\big(\mathrm{ad}^\star_{e_{ij}} e_{ik} + [e_{ij}, e_{ik}], [e_{ij}, e_{ik}]\big) \\ &= \frac{(\lambda_{ij} + \lambda_{jk} - \lambda_{ik})^2}{4\lambda_{jk}} + \lambda_{ik} - \lambda_{jk}. \end{aligned}$$

Similarly we get

$$g\big(R(e_{ij}, e_{jk})e_{jk}, e_{jk}\big) = \frac{(-\lambda_{ij} + \lambda_{jk} - \lambda_{ik})^2}{4\lambda_{ik}} + \lambda_{jk} - \lambda_{ik}.$$

Combining these, we thus have that the k term in (4.5) simplifies to

$$\frac{g\big(R(e_{ij}, e_{ik})e_{ik}, e_{ij}\big)}{\lambda_{ik}} + \frac{g\big(R(e_{ij}, e_{jk})e_{jk}, e_{ij}\big)}{\lambda_{jk}} = \frac{(\lambda_{ij} - \lambda_{ik} + \lambda_{jk})(\lambda_{ij} + \lambda_{ik} - \lambda_{jk})}{2\lambda_{ik}\lambda_{jk}}.$$

The terms with $k<i$ and $i<k<j$ are similar. □

4.2. **Conjugate points along steady geodesics.** Using Theorem 3.5, we show that although the Ricci curvature of the generalized rigid body metric (4.1) can be negative, we still get conjugate points along steady geodesics, just like for the standard rigid body metric. As pointed out in Remark 3.6, some of these conjugate points are not detected by the Misiołek criterion.

Proposition 4.4. *If $u_0 = e_{ij}$ is a steady solution of the Euler equation for the generalized rigid body metric* (4.1)*, then there are conjugate points along the corresponding steady geodesic.*

Proof. According to Lemma 4.1, the basis vectors $e_{k\ell}$ for which $\mathrm{ad}_{e_{ij}}$ is not zero are the ones where $\{k,\ell\} \cap \{i,j\} \neq \emptyset$, i.e. $2m$ vectors in total with $m := n-2$. We can then reorder and rename the basis vectors $\{e_{ij}\}_{1\le i<j\le n}$ into $\{w_k\}_{1\le k\le n(n-1)/2}$ so that

$$[u_0, w_{2j-1}] = w_{2j}, \quad [w_{2j-1}, w_{2j}] = u_0, \quad [w_{2j}, u_0] = w_{2j-1}, \quad 1 \le j \le m,$$

which yields $L(w_{2j-1}) = w_{2j}$, $L(w_{2j}) = -w_{2j-1}$ for $1 \leq j \leq m$, and $L(w_k) = 0$ for all other k. Thus the matrix representation of $L = \mathrm{ad}_{u_0}$ in the basis $\{w_i\}_{1\leq i\leq 2m}$ is a $(2m) \times (2m)$ matrix composed of 2×2 nonzero blocks on the diagonal, all equal to

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and zero everywhere else. Therefore the matrix representation of L^2 is the identity, which obviously commutes with Λ. This proves the existence of conjugate points by Theorem 3.5. □

5. Berger-Cheeger groups

As pointed out by Huizenga and Tapp in [29], all known examples of compact manifolds with positive curvature can be traced back to a Lie group with a bi-invariant metric. Such groups always have non-negative curvature, but bi-invariance is a rather restrictive condition. To obtain a much larger class of examples, one can allow Cheeger deformations of a starting bi-invariant metric along any subgroup as follows.

Definition 5.1. *Let G be a Lie group endowed with a positive-definite bi-invariant metric $\langle\cdot,\cdot\rangle$, and H a subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the respective Lie algebras of G and H. The Cheeger deformation along H is the left-invariant metric g on G given by*

$$g(u, v) = \langle u, v\rangle + \delta\langle Pu, Pv\rangle, \tag{5.1}$$

where $\delta \in (-1,\infty)$ and P is the orthogonal projection onto $\mathfrak{h} \subset \mathfrak{g}$. We denote the associated symmetric operator $\Lambda = I + \delta P$.

Remark 5.2. *It should be noted here that the Cheeger deformation is typically defined in terms of $1/\delta$ rather than δ. Our convention just makes some formulas in this paper easier to write down, but both definitions are virtually the same.*

The first examples of this construction are due to Berger [7], known as Berger spheres. In that case, one shrinks the round metric on $G = S^3$ along the fibers of the Hopf fibration, which correspond to integral curves of the Hopf vector field, so $H = S^1$ and $\delta < 0$.

The curvatures of metrics defined by (5.1) have been explored in some detail. If we shrink the metric along H, which in our notation corresponds to $\delta < 0$, it is known that sectional curvature remains nonnegative, whereas in the expanding case ($\delta > 0$) sectional curvature can become negative (see [29], [49]). The same technique can be used to construct examples of manifolds with positive Ricci curvature [9].

We obtain several new results about metrics of the form (5.1). First, we derive an explicit formula for its geodesics in terms of the Lie group exponential. Using this formula, we develop two new criteria for conjugate points along nonsteady geodesics. The first is Corollary 5.5 of Theorem 3.1 for closed geodesics. The second criterion, Proposition 5.7, is explicitly computable from the initial condition u_0, the subspace H and the deformation parameter δ. Thus, it allows one to see exactly how a metric deformation affects the presence of conjugate points.

As a further application of the calculations mentioned above, we present a new proof of Chavel's description [11] of the conjugate locus of Berger spheres and illustrate how the conjugate locus changes for different values of the deformation parameter δ.

Finally, we show that if the extra condition

$$[\mathfrak{h}^{\perp}, \mathfrak{h}^{\perp}] \subseteq \mathfrak{h} \tag{5.2}$$

is imposed, the Ricci curvature of (5.1) is "block Einstein," which means that it becomes a multiple of the identity when restricted to $\mathfrak{h}$, and a different multiple of the identity when restricted to $\mathfrak{h}^{\perp}$.

An unexpected feature of these metrics is that despite the changes in curvature as δ varies, even to the extent of producing metrics with some negative Ricci curvatures, the behavior of geodesics and conjugate points remains essentially the same.

5.1. Explicit solution of the geodesic equation.

We begin with the solution of the Euler-Arnold equation.

Proposition 5.3. *In the same notation as Definition 5.1, let* $\mathfrak{h}^\perp$ *denote the orthogonal complement of* $\mathfrak{h}$ *in* $\mathfrak{g}$ *under the bi-invariant metric* $\langle\cdot,\cdot\rangle$. *Then*

$$[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h} \qquad \text{and} \qquad [\mathfrak{h},\mathfrak{h}^\perp] \subset \mathfrak{h}^\perp. \tag{5.3}$$

As a consequence, if we write $u(t) = p(t) + q(t)$ *where* $p \in \mathfrak{h}$ *and* $q \in \mathfrak{h}^\perp$, *then the Euler-Arnold equation* (2.13) *for* u *becomes*

$$p'(t) = 0, \qquad \frac{dq}{dt} = \delta \mathrm{ad}_{p(t)} q(t), \tag{5.4}$$

with solution

$$p(t) = p_0, \qquad q(t) = \mathrm{Ad}_{\eta(t)} q_0, \qquad \eta(t) := \exp(\delta t p_0).$$

Proof. The first part of (5.3) is just the definition of a Lie subalgebra. To show the second part, let $u \in \mathfrak{h}$ and $w \in \mathfrak{h}^\perp$: we will show that $\langle \mathrm{ad}_u w, v\rangle = 0$ for any $v \in \mathfrak{h}$. This follows from ad-invariance of the metric and $\mathrm{ad}_u v \in \mathfrak{h}$, since

$$\langle \mathrm{ad}_u w, v\rangle = -\langle w, \mathrm{ad}_u v\rangle = 0.$$

Writing $u = p + q$ where $p \in \mathfrak{h}$ and $q \in \mathfrak{h}^\perp$, we obtain $\Lambda u = (1+\delta)p + q$, so that

$$\mathrm{ad}_u \Lambda u = \mathrm{ad}_{p+q}\big((1+\delta)p + q\big) = \mathrm{ad}_p q + (1+\delta)\mathrm{ad}_q p = -\delta \mathrm{ad}_p q.$$

The Euler-Arnold equation (2.13) thus becomes

$$(1+\delta)p'(t) + q'(t) - \delta \mathrm{ad}_{p(t)} q(t) = 0,$$

and since $\mathrm{ad}_p q \in \mathfrak{h}^\perp$, we obtain the splitting (5.4). Obviously $p(t) = p_0$ solves the first part, and the second part follows from the definition of ad as the derivative of Ad on any group. □

Geodesics can be described explicitly in terms of the group exponential map as follows.

Proposition 5.4. *In the same notation as Definition 5.1, if* $\gamma(t)$ *is a geodesic with* $\gamma(0) = \mathrm{id}$ *and* $\gamma'(0) = u_0 = p_0 + q_0$, *for* $p_0 \in \mathfrak{h}$ *and* $q_0 \in \mathfrak{h}^\perp$, *then*

$$\gamma(t) = e^{t\Lambda u_0} e^{-\delta t p_0}. \tag{5.5}$$

Proof. In Proposition 5.3, we found that the solution $u(t)$ of the Euler-Arnold equation (2.13) was given by

$$u(t) = e^{\delta t p_0}(p_0 + q_0)e^{-\delta t p_0} = \eta(t) u_0 \eta(t)^{-1}.$$

Now using the flow equation $\gamma'(t) = \gamma(t)u(t)$, we find that

$$\begin{aligned}\frac{d}{dt}\big(\gamma(t)\eta(t)\big) &= \gamma'(t)\eta(t) + \gamma(t)\eta'(t) = \gamma(t)\eta(t)u_0\eta(t)^{-1}\eta(t) + \delta\gamma(t)\eta(t)p_0 \\ &= \gamma(t)\eta(t)(u_0 + \delta p_0) = \gamma(t)\eta(t)\Lambda u_0.\end{aligned}$$

Thus since $\gamma(0)\eta(0) = \mathrm{id}$, we must have

$$\gamma(t)\eta(t) = e^{t\Lambda u_0},$$

and formula (5.5) follows using $\eta(t) = e^{\delta t p_0}$. □

Formula (5.4) together with Theorem 3.1 gives a simple criterion for conjugate points along nonsteady geodesics in any Berger-Cheeger group. Essentially, if the initial velocity Λu_0 would yield a closed geodesic under the bi-invariant metric on G, then the initial velocity u_0 yields a geodesic with conjugate points under the Berger-Cheeger metric. Since closed geodesics for bi-invariant metrics on compact Lie groups are common, many examples can be obtained this way.

Corollary 5.5. *Suppose* G *is a Berger-Cheeger group, and* γ *is a nonsteady geodesic with* $\gamma(0) = \mathrm{id}$ *and* $\gamma'(0) = u_0$. *Assume that* $e^{\tau\Lambda u_0} = \mathrm{id}$ *for some* $\tau > 0$. *Then there is a conjugate point along the geodesic* γ *given by* (5.5).

Proof. By the inclusions (5.3), the operators ad_{p_0} for $p_0 \in \mathfrak{h}$ preserve the orthogonal decomposition, and therefore so does $\mathrm{Ad}_{\eta(t)}$ for any t; in particular $\mathrm{Ad}_{\eta(t)}$ commutes with $\Lambda = I + \delta P$. We conclude that $\mathrm{Ad}_{\eta(t)}$ is an isometry of the Berger-Cheeger metric, since it is an isometry of the bi-invariant metric.

By the explicit formula (5.5), we will have $\gamma(\tau) = \eta(\tau)^{-1}$. Thus $\mathrm{Ad}_{\gamma(\tau)}$ preserves the Berger-Cheeger metric as well, and we conclude that right-translation by $\gamma(\tau)$ is an isometry. Thus Theorem 3.1 implies that $\gamma(\tau)$ is conjugate to $\gamma(0)$. □

5.2. **The Jacobi equation and the index form.**

Proposition 5.6. *Suppose G is a Lie group with subgroup H and left-invariant metric defined at the identity by an operator $\Lambda = I + \delta P_{\mathfrak{h}}$ for $\delta > -1$, as in Proposition 5.3, and assume the additional condition (5.2) holds: $[\mathfrak{h}^\perp, \mathfrak{h}^\perp] \subseteq \mathfrak{h}$. Then the Jacobi equation for a Jacobi field $J(t) = y(t) \circ \gamma(t)$ along a geodesic $\gamma(t)$ with $\gamma(0) = \mathrm{id}$ and $\gamma'(0) = p_0 + q_0$ is given by a constant-coefficient system for $\mathrm{Ad}_{\eta(t)} y(t)$, where $\eta(t) = \exp(\delta t p_0)$.*

Proof. As in Lemma 2.8, the Jacobi equation takes the form (2.14). For $\Lambda = \delta P_{\mathfrak{h}} + I$, writing $z(t) = \mathrm{Ad}_{\eta(t)}\big(z_1(t) + z_2(t)\big)$ where $z_1(t) \in \mathfrak{h}$ and $z_2(t) \in \mathfrak{h}^\perp$, we have $\Lambda u(t) = \delta p_0 + u(t) = \mathrm{Ad}_{\eta(t)}\big((1+\delta)p_0 + q_0\big)$ and $\Lambda z(t) = \mathrm{Ad}_{\eta(t)}\big((1+\delta)z_1(t) + z_2(t)\big)$, since $\mathrm{Ad}_{\eta(t)}$ is an isometry and commutes with Λ. Thus the terms in the linearized Euler equation (2.14) become

$$\begin{aligned}\mathrm{ad}_u \Lambda z + \mathrm{ad}_z \Lambda u &= \mathrm{ad}_{\mathrm{Ad}_\eta(\delta p_0 + u_0)} \mathrm{Ad}_\eta\big(\delta z_1 + z_1 + z_2\big) + \mathrm{ad}_{\mathrm{Ad}_\eta(\delta z_1 + z_1 + z_2)} \mathrm{Ad}_\eta\big(\delta p_0 + u_0\big) \\ &= \delta \mathrm{Ad}_\eta\big(\mathrm{ad}_{q_0} z_1 - \mathrm{ad}_{p_0} z_2\big).\end{aligned}$$

Since

$$\Lambda z'(t) = \Lambda \mathrm{Ad}_{\eta(t)}\big(z_1'(t) + \delta \mathrm{ad}_{p_0} z_1(t) + z_2'(t) + \delta \mathrm{ad}_{p_0} z_2(t)\big),$$

we conclude that the linearized Euler equation for $z(t)$ splits into $\mathfrak{h}$ and $\mathfrak{h}^\perp$ components as

$$z_1'(t) + \delta \mathrm{ad}_{p_0} z_1(t) = 0, \qquad \text{and} \qquad z_2'(t) + \delta \mathrm{ad}_{q_0} z_1(t) = 0. \tag{5.6}$$

Similarly for the linearized flow equation in (2.6). Write $y(t) = \mathrm{Ad}_{\eta(t)}\big(y_1(t) + y_2(t)\big)$, where $y_1(t) \in \mathfrak{h}$ and $y_2(t) \in \mathfrak{h}^\perp$ for all time. then the equation (2.14) for $y(t)$ becomes

$$y_1' + y_2' + \delta \mathrm{ad}_{p_0}(y_1 + y_2) + \mathrm{ad}_{p_0 + q_0}(y_1 + y_2) = z_1 + z_2,$$

which splits into components $y_1(t) \in \mathfrak{h}$ and $y_2(t) \in \mathfrak{h}^\perp$ as the constant-coefficient system

$$y_1'(t) + (1+\delta)\mathrm{ad}_{p_0} y_1(t) + \mathrm{ad}_{q_0} y_2(t) = z_1(t), \qquad y_2'(t) + \mathrm{ad}_{q_0} y_1(t) + (1+\delta)\mathrm{ad}_{p_0} y_2(t) = z_2(t).$$

□

We can attempt to solve this constant-coefficient system using the same method as in Theorem 3.4, but we face some difficulties: first of all, the corresponding operators L and F will no longer map the orthogonal complement of u_0 to itself, and so it is possible there is no solution R of the equation. Secondly the algebraic properties of ad_{p_0} and ad_{q_0} as operators on $\mathfrak{h}$ and $\mathfrak{h}^\perp$ need to be considered separately in each case depending on precisely what the Lie algebra is. In principle however the analysis could be easily done in any specific case.

We will instead illustrate how to detect conjugate points using the index form, via a method that also generalizes to apply for nonsteady geodesics on an arbitrary quadratic Lie group (see Appendix A).

Proposition 5.7. *Let $u_0 = p_0 + q_0$ be an initial condition for a geodesic γ, with $p_0 \in \mathfrak{h}$ and $q_0 \in \mathfrak{h}^\perp$ and $[p_0, q_0] \neq 0$. If*

$$\delta |p_0|^2 |q_0|^2 \big|[q_0, [p_0, q_0]]\big|^2 < (1+\delta)\big|[p_0, q_0]\big|^4\big((1+\delta)|p_0|^2 + |q_0|^2\big), \tag{5.7}$$

then $\gamma(\tau)$ is conjugate to $\gamma(0)$ for sufficiently large τ.

Proof. By Lemma 2.8, the index form is given by

$$I(y, y) = \int_0^\tau \langle \Lambda z - \mathrm{ad}_{\Lambda u} y, z\rangle\, dt, \qquad z := \dot{y} + \mathrm{ad}_u y,$$

for fields $y(t)$ vanishing at $t = 0$ and $t = \tau$. We choose a test field of the form

$$y(t) = \mathrm{Ad}_{\eta(t)}\big(f(t)p_0 + g(t)q_0 + h(t)r_0\big), \qquad r_0 := [p_0, q_0] \in \mathfrak{h}^\perp,$$

where again $\eta(t) = \exp(\delta t p_0)$ as in Proposition 5.3.

It is easy to compute that

$$z = \mathrm{Ad}_\eta\Big(\dot{f} p_0 + h[q_0, r_0] + \dot{g} q_0 + \big(\dot{h} + (1+\delta)g - f\big) r_0 + (1+\delta) h [p_0, r_0]\Big),$$

where the components are expressed in the $\mathfrak{h}$ and $\mathfrak{h}^\perp$ parts since $[q_0, r_0] \in \mathfrak{h}$ and $[p_0, r_0] \in \mathfrak{h}^\perp$. We similarly find that

$$\Lambda z - \mathrm{ad}_{\Lambda u} y = \mathrm{Ad}_\eta\Big((1+\delta)\dot{f} p_0 + \delta h [q_0, r_0] + \dot{g} q_0 + \dot{h} r_0\Big).$$

Using the elementary computations

$$\langle p_0, [q_0, r_0]\rangle = |r_0|^2, \qquad \langle q_0, [p_0, r_0]\rangle = -|r_0|^2, \qquad \langle q_0, r_0\rangle = 0, \qquad \langle r_0, [p_0, r_0]\rangle = 0,$$

we obtain

$$\langle z, \Lambda z - \mathrm{ad}_{\Lambda u} y\rangle = (1+\delta)|p_0|^2\big(\dot f + \tfrac{|r_0|^2}{|p_0|^2}h\big)^2 + |q_0|^2\big(\dot g - (1+\delta)\tfrac{|r_0|^2}{|q_0|^2}h\big)^2 + |r_0|^2\dot h^2 - \beta h^2,$$

$$\beta := (1+\delta)\,\tfrac{|r_0|^4}{|p_0|^2} + (1+\delta)^2\,\tfrac{|r_0|^4}{|q_0|^2} - \delta\big|[q_0, r_0]\big|^2.$$

The first two terms will vanish if we choose $\dot f$ and $\dot g$ proportional to h, which works as long as $\int_0^\tau h(t)\,dt = 0$, since we need f and g to vanish at the endpoints. Therefore choosing $h(t) = \sin\big(\frac{2\pi t}{\tau}\big)$ and f and g to be the appropriate time integrals, we have that the integral of h is zero on $[0,\tau]$, and that

$$I(y,y) = \int_0^\tau \langle z, \Lambda z - \mathrm{ad}_{\Lambda u} y\rangle\,dt = \int_0^\tau |r_0|^2\dot h(t)^2 - \beta h(t)^2\,dt = \frac{2\pi^2|r_0|^2}{\tau} - \frac{\beta\tau}{2}.$$

This is negative for sufficiently large τ as long as $\beta > 0$, which is equivalent to (5.7). □

When $-1 < \delta \le 0$, condition (5.7) is satisfied automatically when $[p_0, q_0] \neq 0$ (the nonsteady case), and every such geodesic has a conjugate point. This reflects the fact that the Ricci curvature is strictly positive in this case, and conjugate points are ensured along every geodesic by the Gromoll-Meyer theorem. When $\delta > 0$, some eigenvalues of the Ricci curvature can become negative, and the condition gives new information.
The Zeitlin model. The Zeitlin metric on $SU(3)$ [54] gives a finite-dimensional approximation for the L^2 geometry of the volume-preserving diffeomorphism group, which describes fluid dynamics on a two-dimensional sphere [44]. It can be described as the Berger-Cheeger metric obtained from taking $G = SU(3)$, $H = SO(3)$ and $\delta = -2/3$ in Definition 5.1. In particular, since δ is negative, we immediately obtain the following corollary of Proposition 5.7.

Corollary 5.8. *Every nonsteady geodesic in the $SU(3)$ Zeitlin model has a conjugate point.*

As for the higher-dimensional groups $SU(n)$ in the Zeitlin model, it turns out that they are no longer Berger-Cheeger groups, but are still quadratic groups since $SU(n)$ carries a bi-invariant metric. Thus we can use a similar criterion for general quadratic Lie groups (see next paragraph), but the computations are more complicated. We leave this problem for a future project.
Generalization to any quadratic Lie group. In Appendix A we use a similar technique to obtain a criterion for conjugate points along a nonsteady geodesic in an arbitrary quadratic Lie group, not just one of Berger-Cheeger type. The main idea is that there is a natural three-dimensional orthogonal frame along any nonsteady geodesic in the same way that $\big\{\{p_0, \mathrm{Ad}_{\eta(t)}q_0, \mathrm{Ad}_{\eta(t)}[p_0, q_0]\big\}$ is a natural orthogonal frame in the proof of Proposition 5.7.
Berger spheres. In the special case of Berger spheres, where $\dim(\mathfrak{h}) = 1$, we see that

$$|p_0|^2\big|[q_0, [p_0, q_0]]\big|^2 = \langle p_0, [q_0, [p_0, q_0]]\rangle^2 = \big|[p_0, q_0]\big|^4,$$

so that (5.7) reduces to

$$\delta|q_0|^2 < (1+\delta)\big((1+\delta)|p_0|^2 + |q_0|^2\big),$$

which is satisfied for any δ, and we conclude that every nonsteady geodesic in a Berger sphere has a conjugate point. In fact a direct computation using Proposition 5.6 shows that the solution operator $\Omega(t) := z_0 \mapsto y(t)$ has determinant

$$\det\Omega(t) = \sin(Rt)\Big(-\delta|q_0|^2 Rt\cos(Rt) + (1+\delta)S\sin(Rt)\Big), \tag{5.8}$$

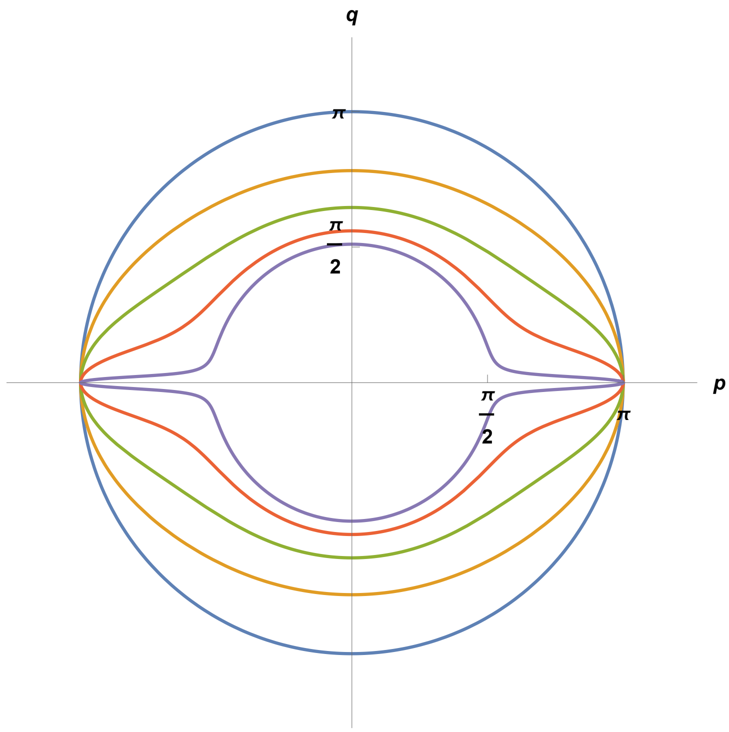
$$R := \sqrt{(1+\delta)^2|p_0|^2 + |q_0|^2}, \qquad S := (1+\delta)|p_0|^2 + |q_0|^2.$$

Note that if $\delta < 0$, the first solution t of $\det\Omega(t) = 0$ will be the unique solution $t \in (\frac{\pi}{2R}, \frac{\pi}{R})$ of the equation

$$\frac{\tan(Rt)}{Rt} = \frac{\delta|q_0|^2}{(1+\delta)\big((1+\delta)|p_0|^2 + |q_0|^2\big)}.$$

Meanwhile if $\delta \ge 0$, then the first solution of $\det\Omega(t) = 0$ is $t = \pi/R$. Hence the conjugarte locus is the image under the exponential map of a topological sphere in $\mathfrak{so}(3)$, a surface of revolution which is an ellipsoid when $\delta \ge 0$ and a more interesting shape when $\delta < 0$. This computation was also done in the typical case $\delta < 0$ by

Chavel [11], using a different method. Below we show two-dimensional slices of the tangent conjugate locus for various values of δ.



Tangent conjugate locus for decreasing values of $\delta = -0.001, -0.25, -0.5, -0.75, -0.95$. The outermost curve corresponds to $\delta = -0.001$ and the innermost one to $\delta = -0.95$.

5.3. **Ricci curvature.** In this section, we compute the Ricci curvature of Berger-Cheeger groups. In the special case that $\mathfrak{h}$ and $\mathfrak{h}^\perp$ factor $\mathfrak{g}$ into a Cartan decomposition, which requires the additional assumption $[\mathfrak{h}^\perp, \mathfrak{h}^\perp] \subseteq \mathfrak{h}$, we get a substantial simplification of the Ricci curvature: the group becomes "block Einstein" in the sense that the Ricci curvature is a constant multiple of the metric when restricted to $\mathfrak{h}$, and a different constant multiple of the metric when restricted to $\mathfrak{h}^\perp$. We focus on this case, and provide a general formula without this assumption at the end – see Remark 5.11.

Proposition 5.9. *Suppose G is a Berger-Cheeger group with subgroup H. Assume that*

$$[\mathfrak{h}^\perp, \mathfrak{h}^\perp] \subseteq \mathfrak{h}. \tag{5.9}$$

Let P denote the orthogonal projection of $\mathfrak{g}$ onto $\mathfrak{h}$ (in the bi-invariant form on $\mathfrak{g}$), and let $Q = I - P$ be the projection onto $\mathfrak{h}^\perp$. If $u \in \mathfrak{h}$ and $v \in \mathfrak{g}$, then the curvature of G under the left-invariant metric (5.1) is given by

$$g(R(u,v)v,u) = \frac{1+\delta}{4}\big|\mathrm{ad}_u P(v)\big|^2 + \frac{(1+\delta)^2}{4}\big|\mathrm{ad}_u Q(v)\big|^2. \tag{5.10}$$

Meanwhile if $u \in \mathfrak{h}^\perp$ and $v \in \mathfrak{g}$, then

$$g(R(u,v)v,u) = \frac{(1+\delta)^2}{4}\left|\mathrm{ad}_u P(v)\right|^2 + \frac{1-3\delta}{4}\left|\mathrm{ad}_u Q(v)\right|^2. \tag{5.11}$$

As before, for any $w \in \mathfrak{g}$, we write $|w|^2 = \langle w, w\rangle$ while $\|w\|^2 = g(w,w) = \langle w, \Lambda w\rangle$.

Proof. We begin with the general formula for sectional curvature on a Lie group with left-invariant metric g as in [38]:

$$g\big(R(u,v)v,u\big) = \tfrac{1}{4}\|\mathrm{ad}^\star_u v + \mathrm{ad}^\star_v u + \mathrm{ad}_u v\|^2 - g(\mathrm{ad}^\star_u v + \mathrm{ad}_u v, \mathrm{ad}_u v) - g(\mathrm{ad}^\star_u u, \mathrm{ad}^\star_v v). \tag{5.12}$$

Recalling the formula (2.12) for $\mathrm{ad}^\star$, and using the formula $\Lambda = I + \delta P$ and its consequence $\Lambda^{-1} = I - \frac{\delta}{1+\delta}P$, we find that for any u and v, we have

$$\mathrm{ad}_u^\star v = -\mathrm{ad}_u v - \delta \mathrm{ad}_u P(v) + \tfrac{\delta}{1+\delta} P(\mathrm{ad}_u v) + \tfrac{\delta^2}{1+\delta} P\big(\mathrm{ad}_u P(v)\big),$$

and we conclude, using the commutator relations (5.3) and (5.9) between $\mathfrak{h}$ and $\mathfrak{h}^\perp$, that

$$\mathrm{ad}_u^\star v + \mathrm{ad}_u v = \delta \left(\mathrm{ad}_{Q(u)} \left(\frac{Q(v)}{1+\delta} - P(v) \right) \right), \tag{5.13}$$

where as expected if $\delta = 0$ the right side is zero since then the metric becomes bi-invariant on $\mathfrak{g}$. If $u \in \mathfrak{h}$, then clearly (5.13) is zero, and together with $\mathrm{ad}_u^\star u = 0$ this shows that the last two terms in (5.12) disappear. Decomposing v into $P(v) + Q(v)$ immediately gives (5.10).

Now suppose that $u \in \mathfrak{h}^\perp$. Using (5.13), after simplifying we get

$$\mathrm{ad}_u^\star v + \mathrm{ad}_u v + \mathrm{ad}_v^\star u = \mathrm{ad}_u Q(v) + (1-\delta)\mathrm{ad}_u P(v),$$
$$g\left(\mathrm{ad}_u^\star v + \mathrm{ad}_u v, \mathrm{ad}_u v\right) = \delta \left(|\mathrm{ad}_u Q(v)|^2 - |\mathrm{ad}_u P(v)|^2 \right).$$

Thus, formula (5.12) becomes

$$\begin{aligned}\langle R(u,v)v, u\rangle &= \frac{1+\delta}{4} |\mathrm{ad}_u Q(v)|^2 + \frac{(1-\delta)^2}{4} |\mathrm{ad}_u P(v)|^2 - \delta \left(|\mathrm{ad}_u Q(v)|^2 - |\mathrm{ad}_u P(v)|^2 \right) \\ &= \frac{1-3\delta}{4} |\mathrm{ad}_u Q(v)|^2 + \frac{(1+\delta)^2}{4} |\mathrm{ad}_u P(v)|^2,\end{aligned}$$

which is (5.11). □

Theorem 5.10. *Suppose G is a compact simple Lie group and H is a compact simple Lie subgroup, with respective Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, and consider the metric g given by (5.1). Then the Ricci curvature of G under g splits into block diagonal form as*

$$\mathrm{Ric}(v,v) = C_1(\delta)|P(v)|^2 + C_2(\delta)|Q(v)|^2,$$

for some constants C_1 and C_2 depending on G, H and the parameter δ.

Proof. In the bi-invariant metric $\langle\cdot,\cdot\rangle$, construct an orthonormal basis $e_1, \ldots, e_m$ of $\mathfrak{h}$ and an orthonormal basis $f_1, \ldots, f_n$ of $\mathfrak{h}^\perp$. Write $v = x + y$, where $x \in \mathfrak{h}$ and $y \in \mathfrak{h}^\perp$. Then, formulas (5.10) and (5.11) become

$$\begin{aligned} g\big(R(e_i,v)v, e_i\big) &= \frac{1+\delta}{4}|\mathrm{ad}_{e_i} x|^2 + \frac{(1+\delta)^2}{4}|\mathrm{ad}_{e_i} y|^2, \qquad 1 \le i \le m, \\ g\big(R(f_j,v)v, f_j\big) &= \frac{(1+\delta)^2}{4} |\mathrm{ad}_{f_j} x|^2 + \frac{1-3\delta}{4} |\mathrm{ad}_{f_j} y|^2, \qquad 1 \le j \le n. \end{aligned} \tag{5.14}$$

Note that the vectors $\{e_1, \ldots, e_m, f_1, \ldots, f_n\}$ form an orthonormal basis in the bi-invariant metric, but not in the g metric; instead the vectors $\{(1+\delta)^{-1/2}e_1, \ldots, (1+\delta)^{-1/2}e_m, f_1, \ldots, f_n\}$ are g-orthonormal. Thus by (5.14), the Ricci curvature is

$$\begin{aligned} \mathrm{Ric}(v,v) &= \frac{1}{1+\delta} \sum_{i=1}^m g\big(R(e_i,v)v, e_i\big) + \sum_{j=1}^n g\big(R(f_j,v)v, f_j\big) \\ &= \frac{1}{4}\sum_{i=1}^m |\mathrm{ad}_{e_i} x|^2 + \frac{1+\delta}{4}\sum_{i=1}^m |\mathrm{ad}_{e_i} y|^2 + \frac{1-3\delta}{4}\sum_{j=1}^n |\mathrm{ad}_{f_j} y|^2 + \frac{(1+\delta)^2}{4}\sum_{j=1}^n |\mathrm{ad}_{f_j} x|^2. \end{aligned} \tag{5.15}$$

We conclude that the Ricci curvature is given for $v = x + y$ by

$$\mathrm{Ric}(v,v) = \mathrm{Ric}(x,x) + \mathrm{Ric}(y,y)$$

since there are no cross-terms in (5.15), and thus it is sufficient to compute them separately. The claim of the theorem now reduces to showing that $\mathrm{Ric}(x,x) = C_1|x|^2$ and $\mathrm{Ric}(y,y) = C_2|y|^2$ for $x \in \mathfrak{h}$ and $y \in \mathfrak{h}^\perp$.

The main principle we will use is that since G is a compact simple Lie group, there is a unique non-degenerate bi-invariant metric on G (up to a constant multiple), and thus there is a positive constant β_G such that for any $v \in \mathfrak{g}$, we have

$$\mathrm{Tr}(\mathrm{ad}_v \mathrm{ad}_v) = -\beta_G |v|^2.$$

See for example [1], Proposition 2.48. This says more explicitly that

$$\sum_{i=1}^{m} |\mathrm{ad}_{e_i} v|^2 + \sum_{j=1}^{n} |\mathrm{ad}_{f_j} v|^2 = \beta_G |v|^2.$$

This formula applies for any v, but we also have a compact simple Lie group H, and the same principle applies to give

$$\sum_{i=1}^{m} |\mathrm{ad}_{e_i} x|^2 = \beta_H |x|^2,$$

where the sum is only taken over $\mathfrak{h}$ and only applied to $x \in \mathfrak{h}$. The Ricci curvature on x is now given using (5.15) by

$$\begin{aligned}
\mathrm{Ric}(x,x) &= \frac{1}{4} \sum_{i=1}^{m} |\mathrm{ad}_{e_i} x|^2 + \frac{(1+\delta)^2}{4} \sum_{j=1}^{n} |\mathrm{ad}_{f_j} x|^2 \\
&= \frac{(1+\delta)^2}{4} \left(\sum_{i=1}^{m} |\mathrm{ad}_{e_i} x|^2 + \sum_{j=1}^{n} |\mathrm{ad}_{f_j} x|^2 \right) - \frac{\delta(2+\delta)}{4} \sum_{i=1}^{m} |\mathrm{ad}_{e_i} x|^2 \\
&= \frac{(1+\delta)^2 \beta_G}{4} |x|^2 - \frac{\delta(2+\delta)\beta_H}{4} |x|^2,
\end{aligned}$$

and we conclude that

$$C_1(\delta) = \frac{(1+\delta)^2 \beta_G - \delta(2+\delta)\beta_H}{4}.$$

The computation for $\mathrm{Ric}(y,y)$ is slightly more complicated since $\mathfrak{h}^\perp$ is not a Lie subalgebra, and thus the sum of terms $|\mathrm{ad}_{e_i} y|^2$ does not represent a Killing form. However we have by (5.15) that

$$\mathrm{Ric}(y,y) = \frac{1+\delta}{4} \sum_{i=1}^{m} |\mathrm{ad}_{e_i} y|^2 + \frac{1-3\delta}{4} \sum_{j=1}^{n} |\mathrm{ad}_{f_j} y|^2. \tag{5.16}$$

Now to compute these two sums, we use the fact that $\mathrm{ad}_{e_i} y \in \mathfrak{h}^\perp$ by (5.3) to write

$$\sum_{i=1}^{m} |\mathrm{ad}_{e_i} y|^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} \langle \mathrm{ad}_{e_i} y, f_j \rangle^2,$$

summing only over the inner products with vectors $f_j \in \mathfrak{h}^\perp$, and similarly since $\mathrm{ad}_{f_j} y \in \mathfrak{h}$ by our additional assumption (5.9),

$$\sum_{j=1}^{n} |\mathrm{ad}_{f_j} y|^2 = \sum_{j=1}^{n} \sum_{i=1}^{m} \langle \mathrm{ad}_{f_j} y, e_i \rangle^2.$$

We now notice that these two sums are exactly the same, since

$$\langle \mathrm{ad}_{f_j} y, e_i \rangle = -\langle y, \mathrm{ad}_{f_j} e_i \rangle = \langle y, \mathrm{ad}_{e_i} f_j \rangle = -\langle \mathrm{ad}_{e_i} y, f_j \rangle,$$

and thus we get

$$\sum_{i=1}^{m} |\mathrm{ad}_{e_i} y|^2 + \sum_{j=1}^{n} |\mathrm{ad}_{f_j} y|^2 = 2 \sum_{j=1}^{n} |\mathrm{ad}_{f_j} y|^2 = \beta_G |y|^2.$$

Formula (5.16) thus becomes

$$\mathrm{Ric}(y,y) = \frac{(1+\delta) + (1-3\delta)}{4} \sum_{j=1}^{n} |\mathrm{ad}_{f_j} y|^2 = \frac{(1-\delta)\beta_G |y|^2}{4},$$

so we conclude

$$C_2(\delta) = \frac{(1-\delta)\beta_G}{4}.$$

□

An interesting class of examples where Theorem 5.10 applies is $G = SU(n)$ and $H = SO(n)$, with the natural inclusion. In this case, $\mathfrak{h}$ consists of antisymmetric real matrices, and since the bi-invariant form is $\langle u, v\rangle = -\frac{1}{2}\mathrm{Tr}(uv)$, this means that $\mathfrak{h}^\perp$ is a subset of the space of symmetric matrices. Therefore, given $v, w \in \mathfrak{h}^\perp$,

$$[v, w]^T = (vw - wv)^T = w^T v^T - v^T w^T = wv - vw = -[v, w],$$

so that $[v, w] \in \mathfrak{h}$. This shows that the condition $[\mathfrak{h}^\perp, \mathfrak{h}^\perp] \subseteq \mathfrak{h}$ is fulfilled. Here the constants are $\beta_H = 2(n^2 - n - 4)$ (using Proposition 4.2) and $\beta_G = 4n$ (from [1], Example 2.50).

Remark 5.11. *If one does not assume the condition $[\mathfrak{h}^\perp, \mathfrak{h}^\perp] \subseteq \mathfrak{h}$, then Proposition 5.9 still holds, except that in the case $u \in \mathfrak{h}^\perp$, the formula becomes*

$$g(R(u, v)v, u) = \frac{1-3\delta}{4}\left|P\big(\mathrm{ad}_u Q(v)\big)\right|^2 + \frac{1}{4}\left|Q(\mathrm{ad}_u \Lambda v)\right|^2.$$

However, this formula alone does not allow us to conclude that the Ricci curvature is block diagonal.

6. Outlook

We have seen that some groups exhibit the property that *every* nonsteady geodesic eventually develops conjugate points. This naturally leads to the question of whether such behavior can be simply characterized. For instance, Berger-Cheeger groups have this property for negative δ in (5.1) (when one shrinks the metric along a subgroup H), but it remains unclear whether the same holds for positive δ. If true, it would suggest that this property persists in a neighborhood of a bi-invariant metric – a kind of stability result (note that a bi-invariant metric possesses only steady geodesics, so this property is vacuously true in that case).

Proposition 5.7 shows that if we fix a particular initial velocity u_0 on the Lie algebra, tangent to a nonsteady geodesic, then for sufficiently small δ depending on u_0 the corresponding geodesic eventually develops conjugate points. However, we do not have a uniform bound on the time it takes for conjugate points to appear.

Another interesting problem is that of the shape of the conjugate locus itself, viewed as a subset of the Lie algebra $\mathfrak{g}$: what does the set $\mathcal{C}$ of all $v \in \mathfrak{g}$ such that $d\exp(v)$ is singular look like? In this work we have shown the conjugate loci of Berger spheres (Section 5). Very few other concrete examples of this set are known, since it is hard to describe it explicitly, but the problem seems more tractable in the context of Lie groups. We know by the work of Warner [53] that an open and dense subset of $\mathcal{C}$ is a smooth hypersurface (codimension one) in $\mathfrak{g}$, but there can be branch points whenever conjugate points of multiplicity higher than one split into two hypersurfaces of lower multiplicity each. What types of branching can occur is also not known (see [17]).

Aside from geodesics and the conjugate locus, it would be interesting to see whether the results on section 4 can be extended to include more general inertia operators – not just those of a generalized rigid body. In particular, the question of whether the Ricci tensor still admits a block decomposition in this more general setting is connected to the study of Ricci flow on Lie groups.

Appendix A. Conjugate points along nonsteady geodesics in a quadratic Lie group

Here we generalize to any quadratic Lie group the method used in Proposition 5.7 to obtain a criterion of conjugacy along nonsteady geodesics in Berger-Cheeger groups.

Let G be a quadratic Lie group, with metric g defined from an operator Λ and a bi-invariant bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ through $g(u, v) = \langle \Lambda u, v\rangle$ (as in (2.11)). Assume $u(t)$ is a nonsteady solution of the Euler-Arnold equation (2.13). Our first task is to construct a special 3-frame along $u(t)$ such that the index form simplifies when computed in the span of this 3-frame. This construction relies on conserved quantities which are direct analogues of the conservation laws for energy and angular momentum in the case of a rigid body, but exist on any quadratic Lie group.

Lemma A.1. *Suppose g is generated by an operator Λ as in* (2.11)*. Then for any nonsteady solution $u(t)$ of the Euler-Arnold equation* (2.13)*, the quantities*

$$k := \langle u(t), \Lambda u(t)\rangle \qquad \textit{and} \qquad \ell := \langle \Lambda u(t), \Lambda u(t)\rangle \tag{A.1}$$

are constant in time. Moreover, the vector fields

$$v_1(t) := u'(t)/g(u'(t), u'(t)), \qquad v_2(t) := k\Lambda u(t) - \ell u(t), \qquad v_3(t) := u(t). \tag{A.2}$$

are mutually orthogonal (though not generally orthonormal) in the metric g.

Proof. Straightforward computations show that quantities k and ℓ are constant. Since $u(t)$ is nonsteady, we know $u'(t)$ is never zero, so v_1 is well-defined. The fact that k and ℓ are constant implies that v_1 is orthogonal to v_3 and v_2. Finally v_2 is orthogonal to v_3 by definitions of k and ℓ. □

To prove the existence of conjugate points, the strategy is to find a variational field y and a time $\tau > 0$ such that $y(0) = y(\tau) = 0$ and the index form at y is negative, as in Proposition 2.5. We now prove a lemma which will be used to establish the negativity of the index form.

Lemma A.2. *Suppose ψ and ϕ are real functions on $[0, \infty)$, bounded below by positive constants. Then for any function $\xi(t)$, and for sufficiently large $\tau > 0$, there is a function $f(t)$ with $f(0) = f(\tau) = 0$, together with $\int_0^\tau f(t)\xi(t)\, dt = 0$, and such that*

$$I := \int_0^\tau \frac{(f')^2}{\psi} - \phi f^2\, dt < 0.$$

Proof. Let $\xi(t)$ be a function and assume $\psi(t) \geq a$ and $\phi(t) \geq b$ for some positive real numbers a, b and all $t \geq 0$. Now choose $f(t) = k_1 \sin \frac{\pi t}{\tau} + k_2 \sin \frac{2\pi t}{\tau}$. Clearly $f(0) = f(\tau) = 0$, and we may obviously choose a nontrivial combination k_1 and k_2 such that $\int_0^\tau f(t)\xi(t)\, dt = 0$. Moreover, we have the upper bound

$$I \leq \int_0^\tau \frac{f'(t)^2}{a} - bf(t)^2\, dt = \frac{\pi^2}{a\tau}\,(k_1^2 + 4k_2^2) - \frac{b\tau}{2}\,(k_1^2 + k_2^2),$$

which will be negative for sufficiently large τ. □

We can now establish our criterion for existence of conjugate points.

Theorem A.3. *Let G be a quadratic Lie group with a left-invariant metric g given by (2.11) for an operator Λ with a bi-invariant nondegenerate bilinear form $\langle \cdot, \cdot \rangle$. Let $u(t)$ be a nonsteady solution of the Euler equation, and let k, ℓ, v_1, and v_2 be defined as in Lemma A.1 Define the following vector fields on the Lie algebra*

$$w := v_1' + \mathrm{ad}_u v_1, \quad x := \Lambda w + \mathrm{ad}_{v_1} \Lambda u,$$

and the following functions $[0, \infty) \to \mathbb{R}$,

$$\psi(t) := \frac{1}{g(v_1, v_1)}, \qquad \phi(t) := \frac{k^2 \langle \Lambda w, \Lambda u \rangle^2}{g(v_2, v_2)} - \langle w, x \rangle.$$

Assume that the functions ψ and ϕ are both bounded below by positive constants. Then there is eventually a conjugate point along the geodesic $\gamma(t)$ corresponding to $u(t)$ via the Euler-Arnold equations (2.2). In particular, γ is not globally minimizing.

Proof. Let $\tau > 0$ and $y : [0, \tau] \to \mathfrak{g}$ be a test field, with decomposition $y(t) = y_1(t)v_1(t) + y_2(t)v_2(t)$ with respect to the fields v_1, v_2 defined by (A.2). We restrict to linear combinations of $v_1(t)$ and $v_2(t)$ because including a term proportional to $v_3(t) = u(t)$ would only produce a larger index form as the reader can check. To compute the index form $I(y, y)$, we need to compute both terms involved in the inner product of the integrand in Equation (2.15). They are given by

$$z = (y_1' - \ell\psi y_2)v_1 + y_2' v_2 + y_1 w,$$
$$\Lambda z + \mathrm{ad}_y \Lambda u = y_1' \Lambda v_1 + y_2' \Lambda v_2 + y_1 x,$$

and by orthogonality of v_1 and v_2, their inner product is

$$\langle z, \Lambda z + \mathrm{ad}_y \Lambda u \rangle = (y_1' - \ell\psi y_2)y_1' \langle v_1, \Lambda v_1 \rangle + (y_2')^2 \langle v_2, \Lambda v_2 \rangle$$
$$+ y_1 \langle (y_1' - \ell\psi y_2)v_1 + y_2' v_2, x \rangle + y_1 \langle w, y_1' \Lambda v_1 + y_2' \Lambda v_2 \rangle + y_1^2 \langle w, x \rangle.$$

Using the following inner products, found after somewhat lengthy but straightforward computations,

$$\langle x, v_1 \rangle = 0, \quad \langle x, v_2 \rangle = k\langle \Lambda w, \Lambda u \rangle - \ell, \quad \langle w, \Lambda v_1 \rangle = 0, \quad \langle w, \Lambda v_2 \rangle = k\langle \Lambda w, \Lambda u \rangle,$$

the integrand of the index form simplifies to

$$\langle z, \Lambda z + \mathrm{ad}_y \Lambda u \rangle = \frac{(y_1')^2}{\psi} + g(v_2, v_2)(y_2')^2 + 2k\langle \Lambda w, \Lambda u \rangle y_1 y_2' + \langle w, x \rangle y_1^2 - \ell(y_2 y_1' + y_1 y_2').$$

The last term attached to ℓ will integrate to zero on $[0,\tau]$ since it is a total time derivative of y_1y_2, which vanishes at $t=0$ and $t=\tau$. Now completing the square in what remains yields

$$\langle z, \Lambda z + \mathrm{ad}_y \Lambda u\rangle = \frac{(y_1')^2}{\psi} + g(v_2,v_2)\left(y_2' + \frac{k\langle \Lambda w, \Lambda u\rangle y_1}{g(v_2,v_2)}\right)^2 + \left(\langle w,x\rangle - \frac{k^2\langle \Lambda w,\Lambda u\rangle^2}{g(v_2,v_2)}\right) y_1^2,$$

plus a term that integrates to zero, and so the index form is given by

$$I(y,y) = \int_0^\tau \frac{(y_1')^2}{\psi} - \phi y_1^2 + g(v_2,v_2)\left(y_2' + \xi y_1\right)^2 \, dt, \tag{A.3}$$

where

$$\xi(t) = \frac{k\langle \Lambda w, \Lambda u\rangle y_1}{g(v_2,v_2)}.$$

Then by Lemma A.2, there exists a function f and a $\tau > 0$ such that $\int_0^\tau \xi(t)f(t)dt = 0$, $f(0) = f(\tau) = 0$ and

$$I := \int_0^\tau \frac{(f')^2}{\psi} - \phi f^2 \, dt < 0. \tag{A.4}$$

Setting $y_1(t) = f(t)$ and $y_2'(t) = -\xi(t)f(t)$, we obtain a field $y = y_1v_1 + y_2v_2$ that vanishes at $t=0$ and $t=\tau$, and for which the index form (A.3) is precisely (A.4). Applying Proposition 2.5 then yields the existence of a conjugate point as claimed. □

In the case of Berger-Cheeger groups as in Section 5, the quantities ϕ and ψ are constant along any geodesic, so the application of the theorem involves only checking that these quantities are positive, which is essentially what we did in Proposition 5.7.

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