A GEOMETRIC RIGIDITY THEOREM FOR HYDRODYNAMICAL BLOWUP

STEPHEN C. PRESTON

Abstract. Suppose there is a smooth solution u of the Euler equation on a 3-dimensional manifold M, with Lagrangian flow η, such that for some Lagrangian path $\eta(t,x)$ and some time T, we have $\int_0^T \lvert\omega(t,\eta(t,x))\rvert\,dt = \infty$. Then in particular smoothness breaks down at time T by the Beale-Kato-Majda criterion. We know by the work of Arnold that the Lagrangian solution is a geodesic in the group of volume-preserving diffeomorphisms.

We show that either there is a sequence $t_n \nearrow T$ such that the corresponding geodesic fails to minimize length on each $[t_n, t_{n+1}]$, or there is a basis $\{e_1, e_2, e_3\}$ of T_xM with e_3 parallel to the initial vorticity vector $\omega_0(x)$ such that the components of the stretching matrix $\Lambda(t,x) = (D\eta(t,x))^{\mathrm{T}} D\eta(t,x)$ satisfy

$$\int_0^T \frac{\Lambda_{33}(\tau,x)\,d\tau}{\Lambda^{11}(\tau,x)+\Lambda^{22}(\tau,x)} < \infty \qquad \text{and} \qquad \lim_{t\to T} \frac{\int_0^t \Lambda^{11}(\tau,x)\,d\tau}{\int_0^t \Lambda^{22}(\tau,x)\,d\tau} = 0.$$

The former possibility can be studied in terms of the two-point minimization approach of Brenier on volume-preserving maps, while the latter gives a precise sense in which the vorticity vector tends to align with the intermediate eigenvector of the stretching matrix Λ.

Consider a three-dimensional ideal fluid flow on a manifold M described by a time-dependent velocity field $u\colon [0,T)\times M \to TM$, satisfying the Euler equation

$$u_t + \nabla_u u = -\nabla p, \qquad \operatorname{div} u \equiv 0, \qquad u(0,x) = u_0(x). \tag{1}$$

Here the pressure is determined implicitly by $\Delta p = -\operatorname{div}(\nabla_u u)$. We assume u_0 is C^∞.

The motion of particles is described by the flow map $\eta\colon [0,T)\times M \to M$, defined by

$$\eta_t(t,x) = u\big(t,\eta(t,x)\big), \qquad \eta(0,x) = x;$$

the incompressibility constraint $\operatorname{div} u = 0$ becomes

$$\det\big(D\eta(t,x)\big) \equiv 1 \text{ or } \eta^*\mu = \mu$$

in terms of the volume form μ. The particle map satisfies the Euler equation

$$\frac{D}{\partial t}\frac{\partial \eta}{\partial t}(t,x) = -\nabla p\big(t,\eta(t,x)\big) \tag{2}$$

with initial conditions $\eta(0,x) = x$ and $\eta_t(0,x) = u_0(x)$.

The vorticity vector field $\omega = \operatorname{curl} u$ satisfies

$$\omega_t + [u,\omega] = 0,$$

which leads to the important conservation law

$$\omega\big(t,\eta(t,x)\big) = D\eta(t,x)\big(\omega_0(x)\big). \tag{3}$$

We can think of η as a geodesic in the volumorphism group

$$\mathcal{D}_\mu(M) = \{\eta \in C^\infty(M,M) \,|\, \eta^*\mu = \mu\},$$

as pointed out by Arnold [A], where the Riemannian metric is generated by the L^2 norm of the velocity field (i.e., the kinetic energy). If $s > \frac{5}{2}$, then the closure of $\mathcal{D}_\mu(M)$ in the Sobolev H^s topology is denoted by $\mathcal{D}^s_\mu(M)$, and it is a smooth submanifold of $H^s(M,M)$; furthermore the geodesic equation is a smooth ODE on this manifold, by results of Ebin-Marsden [EM]. Thus there is a smooth exponential

Date: September 9, 2010.

map $\exp_{\mathrm{id}}$ from a neighborhood of 0 in the space of divergence-free H^s vector fields $T_{\mathrm{id}}\mathcal{D}_\mu^s(M)$ to $\mathcal{D}_\mu^s(M)$ which takes u_0 to $\eta(1)$, with η satisfying (2).

Singular values of the exponential map are called *conjugate points*: those where the differential of the exponential map fails to be injective are *monoconjugate*, while those where the differential fails to be surjective are *epiconjugate* points. (In infinite dimensions, these are generally different; see Grossman [Gr] and Biliotti et al. [BEPT] for an elaboration of these issues.) The derivative of the exponential map at 0 is the identity map, which implies by smoothness and the inverse function theorem that there is an H^s neighborhood of the 0 vector in which the exponential map is nonsingular. If η is a geodesic with $\eta(0) = \mathrm{id}$, and $\eta(a)$ is monoconjugate to $\eta(0)$, then η cannot be minimizing on $[0, b]$ for any $b > a$. Intuitively, a conjugate point represents a family of geodesics all starting at the same point which meet to first order at the endpoint; the most familiar example is the north and south poles on a sphere. (For details, see a text such as Spivak [Sp], do Carmo [dC], or Lang [L].)

Along a geodesic η in $\mathcal{D}_\mu^s(M)$, there are times $b > a > 0$ (either of which may be infinite) such that $\eta(t)$ is not conjugate to $\eta(0)$ for $t < a$, and $\eta(t)$ is epiconjugate to $\eta(0)$ for $a \le t < b$. Furthermore for a countable dense set in $[a, b)$, $\eta(t)$ is monoconjugate to $\eta(0)$. See [P2] for an elaboration of this phenomenon. (In the cases where computations have been done explicitly, a is finite and b is infinite unless the initial velocity field u_0 is harmonic.)

Global existence of solutions to (1) is equivalent to geodesic completeness, which in turn is equivalent to the exponential map being defined on all of $T_{\mathrm{id}}\mathcal{D}_\mu^s(M)$. The maximal time of existence T is finite if and only if the Beale-Kato-Majda criterion [BKM] is satisfied:

$$\int_0^T \sup_{x\in M} |\omega(t,x)|\, dt = \infty. \tag{4}$$

(We should note that strictly speaking, Beale-Kato-Majda only proved this when $M = \mathbb{T}^3$; however it seems likely that it holds for any compact three-dimensional manifold. Everything we will do in what follows works on an arbitrary compact three-dimensional Riemannian manifold.) Several other criteria are known: see Constantin [Co] and Gibbon [Gi] for surveys of the literature, as well as Deng et al. [DHY] and Chae [Ch] for some recent results. However our analysis depends only on assuming a slightly strengthened version of (4).

Assumption 1. There is a point $x \in M$ such that

$$\int_0^T \left|\omega\big(t, \eta(t,x)\big)\right|\, dt = \infty.$$

Clearly Assumption 1 is only possible if $\omega_0(x) \neq 0$, via (3). Hence we can choose an orthonormal basis $\{e_1, e_2, e_3\}$ at T_xM which is oriented in the usual way and such that e_3 is a positive multiple of $\omega_0(x)$. (This basis is of course unique only up to a rotation.) If we define the stretching matrix by

$$\Lambda(t,x) = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{12} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{13} & \Lambda_{23} & \Lambda_{33} \end{pmatrix} \equiv D\eta(t,x)^{\mathrm{T}} D\eta(t,x),$$

then $\Lambda_{33} = |D\eta(e_3)|^2$, so that Assumption 1 is obviously equivalent to

$$|\omega_0(x)| \int_0^T \sqrt{\Lambda_{33}(t,x)}\, dt = \infty.$$

As such, we can define a rescaled time variable by

$$s = |\omega_0(x)| \int_0^t \sqrt{\Lambda_{33}(\tau,x)}\, d\tau. \tag{5}$$

Obviously s goes from 0 to ∞ as t goes from 0 to T.

Now we define the property we want to establish for such a solution.

Definition 2. Suppose η is a geodesic in a Riemannian manifold, defined on a maximal interval $[0,T)$. We say that η has a *conjugate cascade* if there is a strictly increasing sequence of times $t_n \nearrow T$ such that $\eta(t_n)$ is conjugate to $\eta(t_{n+1})$.

It does not matter in this definition whether we use monoconjugate or epiconjugate points. By a result of [BEPT], monoconjugate points are dense in epiconjugate points; hence if we know that for every $a<T$ there is a $b\in(a,T)$ such that $\eta(a)$ is epiconjugate to $\eta(b)$, then we know that for any $\varepsilon>0$, there is a $c\in(b,b+\varepsilon)$ such that $\eta(a)$ is monoconjugate to $\eta(c)$. Since this is the property we will actually demonstrate, the use of "conjugate" presents no ambiguity.

In addition, since the minimum time between conjugate locations is governed by the maximum of the Riemannian sectional curvature, a conjugate cascade implies that the sectional curvature must approach positive infinity in directions containing the geodesic's tangent vector.

It should be noted that a conjugate cascade is very different from the pathological behavior of conjugate points discussed in [EMP] and elaborated in [P1] and [P2]. The difference here is that it is easy to have points conjugate to the *initial time*; it is much harder to find points conjugate to *each previous time*.

The main result of this paper is the following rigidity theorem, which says that among blowup scenarios, the conjugate cascade is "typical." If blowup does not occur, then we get some fairly precise information about the structure of the stretching matrix Λ.

Theorem 3. *Suppose η is a solution of the Euler equation (2) with maximal time of existence $T<\infty$, satisfying Assumption 1 for some x. Then either η experiences a conjugate cascade, or there is a choice of orthonormal basis $\{e_1,e_2,e_3\}$ of T_xM such that Λ satisfies the special conditions*

$$\lim_{t\to T}\frac{\int_0^t \Lambda^{11}(\tau)\,d\tau}{\int_0^t \Lambda^{22}(\tau)\,d\tau}=0 \tag{6}$$

and

$$\int_0^T \frac{\Lambda_{33}(\tau)}{\Lambda^{11}(\tau)+\Lambda^{22}(\tau)}\,d\tau<\infty. \tag{7}$$

(The two possibilities are not necessarily mutually exclusive.)

Roughly speaking (6) and (7) quantify the tendency of vorticity to align with the middle eigenvector of Λ, a well-known numerical observation [GGH].

We will prove this theorem through a sequence of lemmas in the remainder of this section. First we recall the following theorem proved in [P1].

Theorem 4. *Let η be a geodesic on $\mathcal{D}_\mu(M)$. If for some point x, the boundary value problem*

$$\frac{d}{dt}\left(\Lambda(t,x)\frac{dy}{dt}\right)+\omega_0(x)\times\frac{dy}{dt}=0,\qquad y(a)=0,\qquad y(b)=0, \tag{8}$$

has a solution, then $\eta(a)$ and $\eta(b)$ are epiconjugate.

Again, by expanding the times slightly we can get the points to be monoconjugate.

Corollary 5. *The curve η experiences a conjugate cascade if there is a sequence $t_n\nearrow T$ such that for each n, equation (8) has a solution with $a=t_n$ and $b=t_{n+1}$.*

Hence the phenomenon is immediately reduced to the oscillation theory of the simple ODE (8). A system of the form (8) is said to be *oscillatory at $t=T$* if there is a sequence $t_n\nearrow T$ such that there are solutions for $a=t_n$ and $b=t_{n+1}$ for all n. (For this definition we may have $T=\infty$.)

In what remains we will analyze this ODE in great detail. Since from now on everything happens at the point x, we will omit reference to it to simplify notation.

First we reduce the three-dimensional system (8) to a two-dimensional system (in the plane orthogonal to the vorticity vector).

Lemma 6. *Suppose $\Lambda(t)$ is a positive-definite symmetric matrix with $\det \Lambda(t) \equiv 1$, and ω_0 is a positive number. Then the equation*

$$\frac{d}{dt}\left(\begin{bmatrix}\Lambda_{11} & \Lambda_{12} & \Lambda_{13}\\ \Lambda_{12} & \Lambda_{22} & \Lambda_{23}\\ \Lambda_{13} & \Lambda_{23} & \Lambda_{33}\end{bmatrix}\frac{d}{dt}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}\right)+\omega_0\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\frac{d}{dt}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \tag{9}$$

is oscillatory on $[0,T)$ if and only if the reduced equation

$$\frac{d}{dt}\left(\begin{bmatrix}\Lambda^{22}/\Lambda_{33} & -\Lambda^{12}/\Lambda_{33}\\ -\Lambda^{12}/\Lambda_{33} & \Lambda^{11}/\Lambda_{33}\end{bmatrix}\frac{d}{dt}\begin{bmatrix}y_1\\ y_2\end{bmatrix}\right)+\omega_0\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\frac{d}{dt}\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \tag{10}$$

is oscillatory on $[0,T)$, where Λ^{ij} are the components of the inverse of Λ.

Proof. Let $0 \le a < b < T$. By a well-known general principle (see e.g., Reid [Re]), a self-adjoint system of the form (9) or (10) has a solution with two zeroes in $[a,b]$ if and only if there is a continuous and piecewise-differentiable y with $y(a) = y(b) = 0$, not identically zero, such that the corresponding index form $I(y,y)$ is nonpositive.

For (9), the index form is

$$\begin{aligned} I_3(y,y) = \int_a^b \big(&\Lambda_{11}\dot{y}_1^2 + \Lambda_{22}\dot{y}_2^2 + \Lambda_{33}\dot{y}_3^2 + 2\Lambda_{12}\dot{y}_1\dot{y}_2 + 2\Lambda_{13}\dot{y}_1\dot{y}_3 + 2\Lambda_{23}\dot{y}_2\dot{y}_3 \\ &+ \omega_0(y_1\dot{y}_2 - y_2\dot{y}_1)\big)\,dt, \end{aligned} \tag{11}$$

while for (10) the index form is

$$I_2(y,y) = \int_a^b \Big(\tfrac{1}{\Lambda_{33}}\big(\Lambda^{22}\dot{y}_1^2 - 2\Lambda^{12}\dot{y}_1\dot{y}_2 + \Lambda^{11}\dot{y}_2^2\big) + \omega_0(y_1\dot{y}_2 - y_2\dot{y}_1)\Big)\,dt. \tag{12}$$

Completing the square in the first line of (11), we get

$$\begin{aligned} &\int_a^b \big(\Lambda_{11}\dot{y}_1^2 + \Lambda_{22}\dot{y}_2^2 + \Lambda_{33}\dot{y}_3^2 + 2\Lambda_{12}\dot{y}_1\dot{y}_2 + 2\Lambda_{13}\dot{y}_1\dot{y}_3 + 2\Lambda_{23}\dot{y}_2\dot{y}_3\big)\,dt \\ &\quad = \int_a^b \Big[\big(\Lambda_{11} - \tfrac{\Lambda_{13}^2}{\Lambda_{33}}\big)\dot{y}_1^2 + \big(\Lambda_{22} - \tfrac{\Lambda_{23}^2}{\Lambda_{33}}\big)\dot{y}_2^2 + 2\big(\Lambda_{12} - \tfrac{\Lambda_{13}\Lambda_{23}}{\Lambda_{33}}\big)\dot{y}_1\dot{y}_2 + \Lambda_{33}\big(\dot{y}_3 + \tfrac{\Lambda_{12}}{\Lambda_{33}}\dot{y}_1 + \tfrac{\Lambda_{23}}{\Lambda_{33}}\dot{y}_2\big)^2\Big]\,dt. \end{aligned}$$

Now since the determinant of Λ is 1, we easily see that the components of the inverse of Λ are given by $\Lambda^{11} = \Lambda_{22}\Lambda_{33} - \Lambda_{23}^2$, $\Lambda^{22} = \Lambda_{11}\Lambda_{33} - \Lambda_{13}^2$, and $\Lambda^{12} = \Lambda_{13}\Lambda_{23} - \Lambda_{12}\Lambda_{33}$. Therefore we have

$$I_3(y,y) = I_2(y,y) + \int_a^b \Lambda_{33}\left[\dot{y}_3 + \tfrac{\Lambda_{13}}{\Lambda_{33}}\dot{y}_1 + \tfrac{\Lambda_{23}}{\Lambda_{33}}\dot{y}_2\right]^2\,dt. \tag{13}$$

The last term is clearly nonnegative, so that $I_2(y,y) \le I_3(y,y)$; hence (10) is oscillatory if (9) is.

To prove the other direction, observe by Cauchy-Schwarz that

$$\int_a^b \Lambda_{33}\left[\dot{y}_3 + \tfrac{\Lambda_{13}}{\Lambda_{33}}\dot{y}_1 + \tfrac{\Lambda_{23}}{\Lambda_{33}}\dot{y}_2\right]^2\,dt \ge \left(\int_a^b \tfrac{\Lambda_{13}}{\Lambda_{33}}\dot{y}_1 + \tfrac{\Lambda_{23}}{\Lambda_{33}}\dot{y}_2\,dt\right)^2 \Big/ \int_a^b \tfrac{dt}{\Lambda_{33}},$$

using the fact that $y_3(a) = y_3(b) = 0$.

Now if (10) is oscillatory on $[0,T)$, then given any $a \in [0,T)$ we can find a $c > a$ and a nontrivial solution $\tilde{y}$ of (10) on $[a,c]$ with $\tilde{y}(a) = \tilde{y}(c) = 0$. We can also find a $b > c$ and a nontrivial solution $\overline{y}$ on $[c,b]$ with $\overline{y}(c) = \overline{y}(b) = 0$. Extend $\tilde{y}$ and $\overline{y}$ to all of $[a,b]$ by defining $\tilde{y}$ to be zero on $[c,b]$ and $\overline{y}$ to be zero on $[a,c]$. By linearity we can find constants α and β to give a nontrivial linear combination $y := \alpha\tilde{y} + \beta\overline{y}$ such that

$$\int_a^b \left(\dot{y}_1\tfrac{\Lambda_{13}}{\Lambda_{33}} + \dot{y}_2\tfrac{\Lambda_{23}}{\Lambda_{33}}\right)\,dt = 0. \tag{14}$$

We now define a third component for y by

$$y_3 = -\int_a^t \left(\dot{y}_1 \tfrac{\Lambda_{13}}{\Lambda_{33}} + \dot{y}_2 \tfrac{\Lambda_{23}}{\Lambda_{33}}\right) d\tau,$$

and (14) ensures $y_3(a) = y_3(b) = 0$. We have $I_2(\tilde{y}, \tilde{y}) = I_2(\overline{y}, \overline{y}) = 0$ and $I_2(\tilde{y}, \overline{y}) = 0$ on the interval $[a, b]$, so that by (13) we have

$$I_3(y, y) = I_2(y, y) = 0.$$

Hence equation (9) has a solution with at least two zeroes in $[a, b]$. □

Now working with (10), define a new time variable s as in (5). Then the index form (12) takes the form

$$I_2(y, y) = \omega_0 \int_{\overline{a}}^{\overline{b}} \left(\Phi_{11}(\tfrac{dy_1}{ds})^2 + 2\Phi_{12} \tfrac{dy_1}{ds}\tfrac{dy_2}{ds} + \Phi_{22}(\tfrac{dy_2}{ds})^2 + y_1 \tfrac{dy_2}{ds} - y_2 \tfrac{dy_1}{ds}\right) ds, \tag{15}$$

where

$$\Phi_{11} = \frac{\Lambda^{22}}{\sqrt{\Lambda_{33}}}, \quad \Phi_{12} = -\frac{\Lambda^{12}}{\sqrt{\Lambda_{33}}}, \quad \text{and} \quad \Phi_{22} = \frac{\Lambda^{11}}{\sqrt{\Lambda_{33}}}. \tag{16}$$

Observe that since $\det \Lambda = 1$, we have $\Phi_{11}\Phi_{22} - \Phi_{12}^2 \equiv 1$.

Now set $\Phi(s) = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12} & \Phi_{22} \end{bmatrix}$ and $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Obviously the index form (15) is zero on $[\overline{a}, \overline{b}]$ for some y if and only if the solution of the 2×2 matrix equation

$$\frac{d}{ds}\left(\Phi \frac{dY}{ds}\right) + J\frac{dY}{ds} = 0, \qquad Y(\overline{a}) = 0, \qquad Y'(\overline{a}) = \text{id} \tag{17}$$

has a solution with $\det(Y(\bar{b})) = 0$, since we can then find a vector z_0 with $Y(\bar{b})(z_0) = 0$; in that case $y(t) = Y(s(t))(z_0)$ is the desired solution of (10).

This leads to a useful criterion in terms of a first-order 2×2 ODE system.

Lemma 7. *Equation* (17) *is oscillatory at $s = \infty$ if and only if there is a sequence $s_n \nearrow \infty$ such that the solution of the matrix equation*

$$\frac{dW}{ds} + J\Phi(s)^{-1}W(s) = 0 \tag{18}$$

with $W(0) = id$ satisfies

$$Tr\left(W(s_{n+1})W(s_n)^{-1}\right) = 2 \tag{19}$$

for every n.

Proof. We just define $W(s) = \Phi(s)Y'(s)$. Clearly (17) reduces to (18) under this substitution, with $W(s_n) = \Phi(s_n)$. On the other hand if we replace only the term in parentheses from (17) with W, we obtain

$$\frac{dW}{ds} + J\frac{dY}{ds} = 0,$$

from which we obtain

$$W(s_{n+1}) + JY(s_{n+1}) = W(s_n)$$

since $Y(s_n) = 0$. Since

$$\det(-JY(s_{n+1})) = -\det(Y(s_{n+1})) = 0,$$

we see that $\det(W(s_{n+1}) - W(s_n)) = 0$.

Now the general formula

$$\frac{d}{ds}\ln(\det W) = \text{Tr}\left(\frac{dW}{ds}W^{-1}\right)$$

implies that $\det W$ is constant, and since $W(s_n) = \Phi(s_n)$ we see that $\det W(s) \equiv \det \Phi(s_n) = 1$. So our requirement is that $\det(W(s_{n+1})W(s_n)^{-1} - \mathrm{id}) = 0$.

Now for the matrix $A = W(s_{n+1})W(s_n)^{-1}$, we know $\det A = 1$, so that if $\det(A - \mathrm{id}) = 0$, the matrix A must have eigenvalue 1 with multiplicity two, which implies $\mathrm{Tr}\, A = 2$ by the Cayley-Hamilton theorem. This argument can obviously be reversed, so we get a necessary and sufficient condition (19). □

Now the idea of what we want to do next is the following. Express $\Phi(s)$ in terms of its eigenvalues e^{λ} and $e^{-\lambda}$ and eigenbasis $\begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}$ as

$$\Phi = \begin{bmatrix} \cosh\lambda + \sinh\lambda\cos 2\gamma & \sinh\lambda\sin 2\gamma \\ \sinh\lambda\sin 2\gamma & \cosh\lambda - \sinh\lambda\cos 2\gamma \end{bmatrix}$$

and express $W(s)$ as

$$W = \cosh\psi \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} + \sinh\psi \begin{bmatrix} \cos\beta & -\sin\beta \\ -\sin\beta & -\cos\beta \end{bmatrix}.$$

Then equation (18) takes the form

$$\frac{d\alpha}{ds} = \cosh\lambda - \sinh\lambda\tanh\psi\cos(\alpha+\beta+2\gamma), \tag{20}$$

$$\frac{d\beta}{ds} = \cosh\lambda - \sinh\lambda\coth\psi\cos(\alpha+\beta+2\gamma), \tag{21}$$

$$\frac{d\psi}{ds} = -\sinh\lambda\sin(\alpha+\beta+2\gamma). \tag{22}$$

We observe that α is strictly increasing, and since the trace condition (19) translates into

$$\cosh\psi(s_{n+1})\cosh\psi(s_n)\cos[\alpha(s_{n+1}) - \alpha(s_n)] - \sinh\psi(s_{n+1})\sinh\psi(s_n)\cos[\beta(s_{n+1}) - \beta(s_n)] = 1, \tag{23}$$

it is easy to see that $\lim_{s\to\infty}\alpha(s) = \infty$ is sufficient for a conjugate cascade. Hence if a cascade doesn't happen, we must have

$$\int_0^\infty \big[\cosh\lambda - \sinh\lambda\tanh\psi\cos(\alpha+\beta+2\gamma)\big]\, ds < \infty.$$

Roughly speaking, this should imply that $|\lambda| \to \infty$, $|\psi| \to \infty$, and $\alpha+\beta+2\gamma \to n\pi$ for some integer n. Thus both α and β roughly converge, and thus so does γ.

These statements are not quite correct since $\int_0^\infty |f(s)|\, ds < \infty$ does not imply $f(s) \to 0$ as $s \to \infty$, but we can still get convergence in an average sense. The other problem is that the variables γ and β are not well-defined when Φ or W happens to be the identity, while λ and ψ are not smooth at such times. Hence we will work directly with the components of Φ and W, although we always have the trigonometric coordinates in mind.

With that said, we now write

$$\Phi = \begin{bmatrix} p+r & q \\ q & p-r \end{bmatrix} \qquad \text{and} \qquad W = \begin{bmatrix} a+c & b+d \\ d-b & a-c \end{bmatrix}, \tag{24}$$

where a, b, c, d, p, q, r are all smooth functions of the rescaled time parameter s. The conditions $\det\Phi \equiv 1$ and $\det W \equiv 1$ imply

$$p^2 = 1 + q^2 + r^2 \qquad \text{and} \qquad a^2 + b^2 = 1 + c^2 + d^2. \tag{25}$$

Theorem 8. *Suppose η is a geodesic in $\mathcal{D}_\mu(M)$ satisfying Assumption 1 which does not experience a conjugate cascade as $t \nearrow T$.*

Then there is a basis $\{e_1, e_2, e_3\}$ of T_xM such that, in terms of the new time variable s defined by (5), and the variables p, q, r defining Φ in (24), we have

$$\int_0^\infty p - \sqrt{p^2-1}\, ds < \infty \tag{26}$$

as well as

$$\lim_{\Sigma\to\infty} \frac{\int_0^\Sigma \left[\sqrt{q^2+r^2} - r\right] ds}{\int_0^\Sigma \left[\sqrt{q^2+r^2}\right] ds} = 0. \tag{27}$$

Proof. Set

$$f = 2(ad - bc), \qquad g = 2(ac+bd), \qquad h = a^2+b^2+c^2+d^2, \tag{28}$$

where a, b, c, d are the components of W as in (24). Then equation (25) implies

$$h^2 = 1 + f^2 + g^2.$$

We compute that

$$\begin{aligned} \tfrac{da}{ds} &= -pb - qc + rd, & \tfrac{db}{ds} &= pa - rc - qd, \\ \tfrac{dc}{ds} &= -qa - rb + pd, & \tfrac{dd}{ds} &= ra - qb - pc. \end{aligned} \tag{29}$$

Now equation (25) implies that $a^2 + b^2 \neq 0$ for all time, and hence the functions $\rho = \sqrt{a^2+b^2}$ and α defined by $\rho\cos\alpha = a$ and $\rho\sin\alpha = b$ are well-defined and smooth for all time. (We set $\alpha(0) = 0$ to normalize, since $a(0) = 1$ and $b(0) = 0$.)

Equations (29) imply that

$$\frac{d\alpha}{ds} = \frac{a\dot{b} - b\dot{a}}{a^2+b^2} = p - \frac{qf+rg}{2(a^2+b^2)},$$

using (28). Now $2(a^2+b^2) = h+1$, so we get

$$\begin{aligned} \frac{d\alpha}{ds} &= p - \frac{qf+rg}{h+1} = p - \sqrt{p^2-1} + \sqrt{q^2+r^2} - \frac{qf+rg}{h+1} \\ &= \left(p - \sqrt{p^2-1}\right) + \sqrt{q^2+r^2}\left(1 - \sqrt{\tfrac{h-1}{h+1}}\right) + \frac{\sqrt{(q^2+r^2)(f^2+g^2)} - (qf+rg)}{h+1}. \end{aligned} \tag{30}$$

Clearly each term of (30) is nonnegative, with the first and second being strictly positive. So α is strictly increasing. We next want to show that if α approaches infinity, then there is a conjugate cascade.

We compute that the trace condition (19) is equivalent to

$$a(s_n)a(s_{n+1}) + b(s_n)b(s_{n+1}) - c(s_n)c(s_{n+1}) - d(s_n)d(s_{n+1}) = 1,$$

which in terms of ρ and α is equivalent to $F(s_{n+1}) = 1$ where

$$F(s) = \rho(s)\rho(s_n)\cos\big(\alpha(s) - \alpha(s_n)\big) - \langle \phi(s_n), \phi(s)\rangle,$$

with $\phi(s) = (c(s), d(s))$. Observe that

$$|\phi(s)| = \sqrt{c(s)^2 + d(s)^2} = \sqrt{\rho(s)^2 - 1}.$$

Now if $\lim_{s\to\infty} \alpha(s) = +\infty$, then we can choose $s' > s_n$ such that $\alpha(s') = \alpha(s_n) + 2\pi$, and get that $F(s') \geq \rho(s)\rho(s_n) - \sqrt{(\rho(s)^2-1)(\rho(s_n)^2-1)} \geq 1$. We can also choose $s'' > s_n$ such that $\alpha(s'') = \alpha(s_n) + \pi$, so that $F(s'') \leq -\rho(s)\rho(s_n) + \sqrt{(\rho(s)^2-1)(\rho(s_n)^2-1)} \leq -1$. So between s'' and s' there is a solution s_{n+1} of $F(s_{n+1}) = 1$.

Note that the inequalities on F follow from $\sqrt{(1+x^2)(1+y^2)} - xy \geq 1$, which follows from the Cauchy-Schwarz inequality in the form $1 + 2xy + x^2y^2 \leq 1 + x^2 + y^2 + x^2y^2$.

Hence if η does not experience a conjugate cascade, then $\lim_{s\to\infty} \alpha(s) < \infty$, which implies that each of the three nonnegative terms in (30) has a finite integral. Thus we get (26), in addition to

$$\int_0^\infty \sqrt{q^2+r^2}\left(1 - \sqrt{\frac{h-1}{h+1}}\right) ds < \infty \tag{31}$$

and

$$\int_0^\infty \frac{1}{h+1}\left(\sqrt{(q^2+r^2)(f^2+g^2)} - (qf+rg)\right) ds < \infty. \tag{32}$$

We first explore the consequences of (31). Define $\lambda(s) \geq 0$ by $\sqrt{p^2-1} = \sinh\lambda$. Then λ is certainly continuous, and we have $p = \cosh\lambda$, so that (26) says that

$$\int_0^\infty e^{-\lambda(s)}\, ds < \infty. \tag{33}$$

Hence (31) says that

$$\int_0^\infty (e^\lambda - e^{-\lambda}) u\, ds < \infty,$$

where $u = 1 - \sqrt{\frac{h-1}{h+1}}$. Now since $0 < u \leq 1$, we already know that $\int_0^\infty e^{-\lambda} u\, ds < \infty$, so (31) really tells us that $\int_0^\infty e^\lambda u\, ds < \infty$. Now (33) implies that $\int_0^\infty e^{\lambda(s)}\, ds = \infty$, so if we define a new rescaled time variable by $d\sigma = e^{\lambda(s)}\, ds$, then $\sigma \to \infty$ as $s \to \infty$. So (31) can be written as

$$\int_0^\infty u(\sigma)\, d\sigma < \infty. \tag{34}$$

We can compute using (29) that $\frac{dh}{ds} = 2(rf - qg)$, so that using $h^2 - 1 = f^2 + g^2$ we get

$$\left|\frac{du}{d\sigma}\right| = |2u - u^2| \frac{|rf - qg|}{\sqrt{(f^2+g^2)(q^2+r^2)}} \leq 1. \tag{35}$$

It is easy to see that (34) and (35) together imply that $\lim_{\sigma\to\infty} u(\sigma) \searrow 0$, and hence that $\lim_{s\to\infty} u(s) \searrow 0$. Since $h(s) = \frac{1}{u(s)} + \frac{1}{2-u(s)} - 1$, we see that $\lim_{s\to\infty} h(s) = +\infty$.

Finally we look at the consequence of (32). Our method is inspired by Hartman [H], Lemma 7.1. Since $\lim_{s\to\infty} h(s) = \infty$, there is some L such that $h(s) > 1$ for $s > L$; in particular $f^2 + g^2 > 0$, so that it makes sense to define ξ for $s > L$ by $\frac{f}{\sqrt{f^2+g^2}} = \cos\xi$ and $\frac{g}{\sqrt{f^2+g^2}} = \sin\xi$, and this ξ is smooth on (L, ∞). It is easy to compute that

$$\frac{d\xi}{ds} = 2p - \frac{2h(qf+rg)}{h^2-1} = 2p - \frac{2h}{\sqrt{h^2-1}}(q\overline{f} + r\overline{g}),$$

where $\overline{f} = \frac{f}{\sqrt{f^2+g^2}}$ and $\overline{g} = \frac{g}{\sqrt{f^2+g^2}}$. Now we have

$$\frac{d\xi}{ds} \leq 2\big[p - \sqrt{p^2-1}\big] + 2\left[\Big(\frac{h}{\sqrt{h^2-1}} - 1\Big)(q\overline{f} + r\overline{g})\right] + \left[\sqrt{q^2+r^2} - (q\overline{f} + r\overline{g})\right],$$

and we know that each of the terms is (absolutely) integrable. Thus $\lim_{s\to\infty} \xi(s)$ exists. In particular $\lim_{s\to\infty} \overline{f}(s)$ and $\lim_{s\to\infty} \overline{g}(s)$ exist. Now we never specified the specific directions of $\{e_1, e_2\}$ (the basis of the orthogonal complement of $\omega_0(x)$), so we might as well now rotate them so that $\lim_{s\to\infty} \overline{f}(s) = 0$ and $\lim_{s\to\infty} \overline{g}(s) = 1$.

Now from (32), using (31), we can easily show that

$$\int_L^\infty \left[\sqrt{q^2+r^2} - (q\overline{f} + r\overline{g})\right] ds < \infty. \tag{36}$$

Now write for any $\Sigma > L$

$$\frac{\int_L^\Sigma \left[\sqrt{q^2+r^2} - (q\overline{f} + r\overline{g})\right] ds}{\int_L^\Sigma \sqrt{q^2+r^2}\, ds} + \frac{\int_L^\Sigma \left[q\overline{f} + r(\overline{g}-1)\right] ds}{\int_L^\Sigma \sqrt{q^2+r^2}\, ds} = \frac{\int_L^\Sigma \left[\sqrt{q^2+r^2} - r\right] ds}{\int_L^\Sigma \sqrt{q^2+r^2}\, ds}.$$

Since $\sqrt{q^2+r^2} = \sinh\lambda$ and $\int_0^\infty e^\lambda\, ds = \infty$, we know the denominator approaches infinity as $\Sigma \to \infty$. The second quotient approaches zero as $\Sigma \to \infty$ since $\overline{f} \to 0$ and $\overline{g} \to 1$, while the first quotient approaches zero because of (36). We therefore conclude that

$$\lim_{\Sigma\to 0} \frac{\int_L^\Sigma \left[\sqrt{q^2+r^2} - r\right] ds}{\int_L^\Sigma \sqrt{q^2+r^2}\, ds} = 0.$$

Because of this, and because $\lim_{\Sigma\to\infty} \int_0^\Sigma \sqrt{q^2+r^2}\, ds = \infty$, we also have (27). □

Finally we translate the results of Theorem 8 into the original time variable and in terms of more easily measured quantities.

Corollary 9. *Suppose η is a geodesic in $\mathcal{D}_\mu(M)$ satisfying Assumption 1. If η does not experience a conjugate cascade, then there is an oriented orthonormal basis $\{e_1, e_2, e_3\}$ of T_xM such that $\omega_0(x)$ is parallel to e_3, and the components Λ_{ij} of $\Lambda = (D\eta)^T(D\eta)$ and the components Λ^{ij} of its inverse satisfy the following:*

$$\int_0^T \frac{\Lambda_{33}(\tau)}{\Lambda^{11}(\tau) + \Lambda^{22}(\tau)}\, d\tau < \infty \tag{37}$$

and

$$\lim_{t\to T} \frac{\int_0^t \Lambda^{11}(\tau)\, d\tau}{\int_0^t \Lambda^{22}(\tau)\, d\tau} = 0. \tag{38}$$

Proof. By (16) and (24), we have

$$p = \frac{\Lambda^{11} + \Lambda^{22}}{2\sqrt{\Lambda_{33}}},$$

and using $ds = \omega_0\sqrt{\Lambda_{33}}\, dt$, the inequality (26) becomes

$$\int_0^T \left(\Lambda^{11} + \Lambda^{22} - \sqrt{(\Lambda^{11}+\Lambda^{22})^2 - 4\Lambda_{33}}\right) dt < \infty.$$

Rationalizing, we get

$$\int_0^T \frac{\Lambda_{33}}{\Lambda^{11} + \Lambda^{22} + \sqrt{(\Lambda^{11}+\Lambda^{22})^2 - 4\Lambda_{33}}}\, dt < \infty,$$

so that in particular the integral (37) is finite.

We can replace the term $\sqrt{q^2+r^2}$ appearing in (27) with p, since the integral of $p - \sqrt{q^2+r^2}$ is finite by (26). So (27) becomes

$$\lim_{\Sigma\to\infty} \frac{\int_0^\Sigma (p-r)\, ds}{\int_0^\Sigma (p-r) + (p+r)\, ds} = 0.$$

The only way this can happen is if

$$\lim_{\Sigma\to\infty} \frac{\int_0^\Sigma (p-r)\,ds}{\int_0^\Sigma (p+r)\,ds} = 0,$$

which after changing the time variable is equivalent to (38). □

Notice that a conjugate cascade is closely related to the uniqueness problem for minimizers in the space of volume-preserving maps (VPM). The general minimization problem on VPM was proposed by Brenier [B1] and has been studied further by him [B2], Shnirelman [Sh2], and Ambrosio-Figalli [AF]. The space VPM is the closure in the L^2 topology of the volumorphism group, and is a metric space in this topology. If we think of VPM as a closed subset of the space of all L^2 maps from M^3 to itself, we can find a Hölder-type bound on the intrinsic distance in VPM in terms of the extrinsic L^2 distance by $d_{\mathrm{VPM}}(\eta,\xi) \le C(d_{L^2}(\eta,\xi))^\alpha$ for some $\alpha < 1$. (These results were proved by Shnirelman [Sh1].) A local uniqueness result for minimizing paths in the L^2 topology of VPM would immediately preclude a conjugate cascade, since the intrinsic L^2 distance between successive conjugate point locations goes to zero by the above Hölder estimate, while the geodesic is non-minimizing on any interval $[b,T]$ for $b < T$. As of this writing, such a local uniqueness result is unknown.

References

A. V. Arnold, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluids parfaits (On the differential geometry of infinite-dimensional Lie groups and its applications to the hydrodynamics of perfect fluids)*, Ann. Inst. Grenoble **16** no. 1 (1966), 319–361.

AF. L. Ambrosio and A. Figalli, *Geodesics in the space of measure-preserving maps and plans*, Archive Rat. Mech. Anal. **194** no. 2 (2009), 421–462.

BEPT. L. Biliotti, R. Exel, P. Piccione, and D.V. Tausk, *On the singularities of the exponential map in infinite dimensional Riemannian manifolds,* Math. Ann. **336** no. 2 (2006), 247–267.

BKM. J.T. Beale, T. Kato, and A. Majda, *Remarks on the breakdown of smooth solutions for the 3- D Euler equations*, Comm. Math. Phys. **94** no. 1 (1984), 61–66.

B1. Y. Brenier, *The least action principle and the related concept of generalized flows for incompressible perfect fluids*, J. Amer. Math. Soc. **2** no. 2 (1989), 225–255.

B2. Y. Brenier, *Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations*, Comm. Pure Appl. Math. **52** no. 4 (1999), 411–452.

Ch. D. Chae, *The geometric approaches to the possible singularities in the inviscid fluid flows*, J. Phys. A **41** no. 36 (2008) 365501.

Co. P. Constantin, *Singular, weak and absent: solutions of the Euler equations*, Phys. D **237** no. 14–17 (2008), 1926–1931.

DHY. J. Deng, T. Hou, and X. Yu, *Improved geometric conditions for non-blowup of the 3D incompressible Euler equation*, Comm. PDE **31** no. 1–3 (2006), 293–306.

dC. M.F. do Carmo, *Riemannian geometry*, Birkhäuser, Boston 1992.

EM. D. Ebin and J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. Math. **92** no. 1 (1970), 102–163.

EMP. D. Ebin, G. Misiołek, and S.C. Preston, *Singularities of the exponential map on the volume-preserving diffeomorphism group*, Geom. Funct. Anal. **16** no. 3 (2006), 850–868.

GGH. B. Galanti, J.D. Gibbon, and M. Heritage, *Vorticity alignment results for the three-dimensional Euler and Navier-Stokes equations*, Nonlinearity **10** no. 6 (1997), 1675–1694.

Gi. J.D. Gibbon, *The three-dimensional Euler equations: where do we stand?*, Phys. D **237** no. 14–17 (2008), 1894–1904.

Gr. N. Grossman, *Hilbert manifolds without epiconjugate points*, Proc. Amer. Math. Soc. **16** no. 6 (1965), 1365–1371.

H. P. Hartman, *Ordinary differential equations*, Birkhäuser, Boston 1982.

L. S. Lang, *Differential and Riemannian manifolds*, Springer-Verlag, New York 1995.

P1. S.C. Preston, *On the volumorphism group, the first conjugate point is always the hardest*, Comm. Math. Phys. **267** no. 2 (2006), 493–513.

P2. S.C. Preston, *The WKB method for conjugate points in the volumorphism group*, Indiana U. Math. J. **57** no. 7 (2008), 3303–3327.

Re. W.T. Reid, *Sturmian theory for ordinary differential equations*, Springer-Verlag, New York 1980.

Sh1. A. Shnirelman, *The geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid*, Math. USSR Sbornik **56** no. 1 (1987), 82–109.

Sh2. A. Shnirelman, *Generalized fluid flows, their approximation and applications*, Geom. Funct. Anal. **4** no. 5 (1994), 586–620.

Sp. M. Spivak, *A comprehensive introduction to differential geometry, volume 4*, Publish or Perish, Houston, TX 1999.

Department of Mathematics, University of Colorado, Boulder, CO 80309-0395

E-mail address: Stephen.Preston@colorado.edu