ON THE VOLUMORPHISM GROUP, THE FIRST CONJUGATE POINT IS ALWAYS THE HARDEST

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Abstract. We find a simple local criterion for the existence of conjugate points on the group of volume-preserving diffeomorphisms of a 3-manifold with the Riemannian metric of ideal fluid mechanics, in terms of an ordinary differential equation along each Lagrangian path. Using this criterion, we prove that the first conjugate point along a geodesic in this group is always pathological: the differential of the exponential map always fails to be Fredholm.

1. Introduction

The theory of ideal (inviscid) incompressible fluid mechanics is one of the most mathematically beautiful theories in physics. This is partly because one does not need any parameters to describe the system: as soon as one has a compact Riemannian manifold M, possibly with boundary, one can construct the volume-preserving diffeomorphism group $\mathcal{D}_\mu(M)$, and on this infinite-dimensional manifold define a Riemannian metric using the kinetic energy integral. Arnold [A] showed that the geodesics of this metric on $\mathcal{D}_\mu(M)$ are precisely the ideal incompressible fluid flows on M, in the Lagrangian coordinate description. Thus, once one gets past the fairly serious technical issues of functional analysis involved in constructing a topology on $\mathcal{D}_\mu(M)$ and proving that the geodesic equations are well-posed (as accomplished by Ebin and Marsden [EMa]), one has essentially reduced much of ideal fluid mechanics to a study of geometry. Of course, this does not automatically solve the outstanding problems of fluid mechanics, but it does give a different context to them.

For example, the most significant open problem of ideal fluid dynamics is global existence of solutions in a 3-manifold. From the geometric point of view, this is precisely geodesic completeness. We may thus hope that a better understanding of the geometry of $\mathcal{D}_\mu(M)$ may help in clarifying this problem, or in suggesting new techniques for its solution. Researchers are only beginning to develop this subject, although much progress has been made in recent years, and it is likely to remain an interesting field of study for some time to come.

Conjugate points on $\mathcal{D}_\mu(M)$ have been of interest ever since Arnold [A] computed the sectional curvature on $\mathcal{D}_\mu(\mathbb{T}^2)$, found that it was usually negative but sometimes positive, and asked whether one could find conjugate points. Computational difficulties

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prevented much progress in this direction, until Misiołek [M1] proved that one could construct some simple examples of conjugate points in $\mathcal{D}_\mu(S^3)$ along geodesics corresponding to rigid rotations of the 3-sphere. Here, positive curvature on the underlying manifold M helps one obtain positive curvature on $\mathcal{D}_\mu(M)$, which leads to the conjugate points. More surprisingly (and far more difficult computationally), Misiołek [M2] showed that conjugate points exist on $\mathcal{D}_\mu(\mathbb{T}^2)$, using an example similar to the one about which Arnold had asked.

Since the work of Misiołek, substantial progress has been made in understanding conjugate points on $\mathcal{D}_\mu(M)$. For example, Shnirelman [Sh] proved that the diameter of $\mathcal{D}_\mu(M^3)$ is finite for any 3-D manifold M, and using this result and the generalized flows of Brenier [B], showed that there must be "local cut points" along any sufficiently long geodesic in $\mathcal{D}_\mu(M^3)$: that is, there is an arbitrarily close path joining the two points which is strictly shorter. In finite dimensions, such points must be conjugate; on the infinite-dimensional manifold $\mathcal{D}_\mu(M^3)$, this is not necessarily true. No such result is possible in the 2-D case, since the diameter of $\mathcal{D}_\mu(M^2)$ is infinite.

More recently, Ebin, Misiołek, and the author [EMP] studied the nature of the differential of the exponential map. Singularities of $d\exp$ are precisely the conjugate points, so the nature of conjugate points tells us much about the structure of the exponential map. The map $d\exp$ is a mapping from one infinite-dimensional space to another, and its singularities may be of two types: failure to be injective, and failure to be surjective. (For finite-dimensional mappings, both types always coincide.) Grossman [G] called these singularities monoconjugate points and epiconjugate points, respectively. The authors of [EMP] showed that in $\mathcal{D}_\mu(M^2)$, both types of conjugate points coincide and are of finite order, because the exponential map is Fredholm.

On the other hand, [EMP] also showed that in $\mathcal{D}_\mu(M^3)$, it is possible to have an epiconjugate point that is not a monoconjugate point. Their explicit example is the solid flat torus $D^2 \times S^1$, where the geodesic η is rigid, unit speed rotation of the disc. Here $\eta(\pi)$ is the first conjugate point; it is epiconjugate but not monoconjugate. In addition, for every $\varepsilon > \pi$, there is a $t_o \in (\pi, \pi + \varepsilon)$ such that $\eta(t_o)$ is monoconjugate to $\eta(0)$. The present research will demonstrate that this phenomenon is actually quite typical on three-manifolds. So the structure of conjugate points on $\mathcal{D}_\mu(M^3)$ is in general much more complicated than on $\mathcal{D}_\mu(M^2)$ or on a finite-dimensional Riemannian manifold.

In Section 3, we explain why it has often been easier to find conjugate points in three dimensions than in two. It turns out that one can construct local approximations of Jacobi fields supported near any point in three dimensions, and use these to search for genuine Jacobi fields. In Theorem 3.1, we prove that one can construct a divergence-free vector field in a neighborhood of any Lagrangian path interior to M^3, such that the index form along a geodesic in $\mathcal{D}_\mu(M^3)$ can be approximated by a corresponding index form along this path. (This construction is not possible on a two-dimensional manifold.) The approximate index form comes from the following simple equation along a Lagrangian path:

$$\frac{D^2 y}{dt^2} + R(y, \dot\eta)\dot\eta + \nabla_y \nabla p = 0. \tag{1.1}$$

Here p is the pressure function, and the path $\eta(t)(x)$ in the interior of M satisfies the Lagrangian form of the ideal fluid equation:

$$\frac{D}{dt}\dot\eta = -\nabla p. \tag{1.2}$$

Equation (1.1) is simply the linearization in M^3 of the equation (1.2). If equation (1.1) has a solution $y(t)$ vanishing at times $t = 0$ and $t = a$, then the geodesic η has a Jacobi field vanishing at $t = 0$ and some $t = b$, with b arbitrarily close to a.

The criterion of Theorem 3.1 yields a very simple condition for conjugate points, which is easiest to apply when we are dealing with a steady solution X of the 3-D Euler equations. We find that for any steady solution X with a certain type of fixed point at some x (for example, an elliptic fixed point), there must be a monoconjugate point somewhere along the geodesic, and we can compute its location in terms of $\Delta p(x)$. This is a purely three-dimensional phenomenon: in two dimensions, there are many steady flows that have elliptic fixed points but do not have any conjugate points along the corresponding geodesic, because the curvature operator is nonpositive in all directions. See [P2] for details.

Although Theorem 3.1 applies only in three (or possibly higher) dimensions, it has a sort of converse that is true in dimension two or higher. This converse, Proposition 3.6, states that if a geodesic has a monoconjugate point at $\eta(a)$ for some $t = a$, then along some Lagrangian path, the equation (1.1) must have a solution vanishing at $t = 0$ and some $t = \alpha \le a$. We apply this for some simple two-dimensional flows (rotational fields on rotationally-symmetric surfaces) and obtain a new criterion for them not to have monoconjugate points. A previous result of the author [P2] gives a different criterion, and we show that the two criteria are distinct with examples.

The results of Theorem 3.1 and Proposition 3.6 are quite reminiscent of the results of Friedlander and Vishik [FV], though the method of proof is very different. They found ordinary differential equations such that exponential growth of their solutions implies exponential growth of linearized Euler perturbations, and hence of Jacobi fields along geodesics. Our result on conjugate points is loosely related to positive curvature on $\mathcal{D}_\mu$, while their result on exponential growth of Jacobi fields is loosely related to negative curvature on $\mathcal{D}_\mu$. But although the connection between curvature and Jacobi fields is subtle (as discussed in [P1]), both results show that the most important features of Jacobi fields in $\mathcal{D}_\mu(M^3)$ are determined by certain ordinary differential equations along an arbitrary Lagrangian path.

Theorem 3.1 implies that the first conjugate point along a geodesic in $\mathcal{D}_\mu(M^3)$ is always pathological: we prove in Theorem 4.3 that the differential of the exponential map has either infinite-dimensional kernel or does not have closed range in the L^2 topology. (If M is a surface, the differential of the exponential map always has both finite-dimensional kernel and closed range, by the Fredholmness result of [EMP].

This is a very striking and surprising difference between two-dimensional and three-dimensional fluid mechanics.) A complete classification of the possible behaviors at the first conjugate point would be quite interesting.

Finally we give an example of a geodesic in $\mathcal{D}_\mu(M^3)$ such that monoconjugate points are all of infinite order and dense in an interval. Misiołek [M1] showed that if X is a unit-length left-invariant vector field on S^3, then the corresponding geodesic η has a Jacobi field vanishing at $t = 0$ and $t = \pi$. We compute all of the monoconjugate point locations (using a basis of curl eigenfields on S^3 constructed by Jason Cantarella), and find that they occur at all rational multiples of π greater than or equal to π itself. Furthermore, each one has infinite order. By a result of Biliotti et al. [BEPT], we conclude that for every $\tau \in [\pi, \infty)$, the point $\eta(\tau)$ is epiconjugate to $\eta(0)$. This example and the one in [EMP] give us explicit and natural examples of the sorts of pathological conjugate points first described by Grossman [G] in infinite-dimensional geometry, and recently explored in more depth by [BEPT].

2. Background

In this section, we briefly review the geometry of the volumorphism group $\mathcal{D}_\mu(M)$. Many of the formulas provided here for covariant derivatives, curvature operators, and the index form were derived in [M1], [P1], and [EMP]. We will confine ourselves to the C^∞ case, even though for some technical proofs it is more convenient to use the Sobolev H^s spaces. See Ebin and Marsden [EMa] for the precise constructions.

The space of volumorphisms $\mathcal{D}_\mu(M)$ of a Riemannian manifold M (possibly with boundary ∂M) consists of those C^∞ diffeomorphisms η satisfying $\eta^*\mu = \mu$, where μ is the Riemannian volume form. This space has the structure of a Fréchet manifold. Its tangent spaces $T_\eta\mathcal{D}_\mu(M)$ consist of elements of the form $X\circ\eta$, where X is a vector field on M that is divergence-free and tangent to the boundary. The L^2 Riemannian metric $\langle\langle\cdot,\cdot\rangle\rangle$ on $\mathcal{D}_\mu(M)$ is defined in terms of the metric $\langle\cdot,\cdot\rangle$ on M by the formula

$$\langle\langle U\circ\eta, V\circ\eta\rangle\rangle = \int_M \langle U, V\rangle\circ\eta\ \mu. \tag{2.3}$$

$\mathcal{D}_\mu(M)$ also has a Lie group structure, where the group operator is composition. The differentials of the translation operators at the identity are

$$dL_\eta(X) = D\eta(X) \qquad \text{and} \qquad dR_\eta(X) = X\circ\eta. \tag{2.4}$$

By the change of variables formula for integrals, and the fact that each η is volume-preserving, we see that the metric (2.3) is right-invariant. It is not, however, left-invariant. In some sense, then, all the geometric information about $\mathcal{D}_\mu(M)$ is contained in the left-translations.

To compute covariant derivatives in the metric (2.3), we use the Weyl decomposition of vector fields. This decomposition allows us to write any vector field X on a manifold M as

$$X = U + \nabla f,$$

where U is divergence-free and tangent to ∂M. We construct this decomposition by solving the Neumann problem

$$\Delta f = \operatorname{div} X, \qquad \langle \nabla f, \nu\rangle_{\partial M} = \langle X, \nu\rangle_{\partial M} \tag{2.5}$$

for f, then defining $U := X - \nabla f$. (Here ν is the unit normal on ∂M.) This decomposition is orthogonal in the L^2 metric (2.3). We will denote the orthogonal projections by

$$P(X) = U \qquad \text{and} \qquad Q(X) = \nabla f. \tag{2.6}$$

By right-invariance of the metric, the orthogonal projections in $T_\eta\mathcal{D}_\mu(M)$ are given by

$$P_\eta = dR_\eta \circ P \circ dR_{\eta^{-1}} \qquad \text{and} \qquad Q_\eta = dR_\eta \circ Q \circ dR_{\eta^{-1}}. \tag{2.7}$$

Now we consider covariant derivatives.

Proposition 2.1. *Suppose $\eta(t)$ is a curve in $\mathcal{D}_\mu(M)$, and $J(t)$ is a vector field along $\eta(t)$. Let $X(t)$ be the Eulerian velocity field of η, defined by the formula*

$$X(t) = dR_{\eta(t)^{-1}}\Big(\frac{d\eta}{dt}\Big) = \frac{\partial \eta}{\partial t} \circ \eta(t)^{-1}. \tag{2.8}$$

If we right-translate back to the identity to obtain $Y(t) = dR_{\eta(t)^{-1}}\big(J(t)\big)$, the covariant derivative can be computed using

$$\frac{\widetilde{\mathbf{D}}J}{dt} = dR_{\eta(t)}\left(\frac{\partial Y}{\partial t} + P\big(\nabla_{X(t)}Y(t)\big)\right). \tag{2.9}$$

Proof. Formula (2.9) is a consequence of the formula

$$\frac{DJ}{dt}(t,x) = \frac{\partial Y}{\partial t}\big(t, \eta(t,x)\big) + \nabla_{X(t,\eta(t,x))}Y, \tag{2.10}$$

where the covariant derivative of J is computed along each path $t \mapsto \eta(t,x)$; this just comes from the Chain Rule on M. Projecting both sides of (2.10) onto $T_\eta\mathcal{D}_\mu(M)$, we obtain (2.9). A more detailed derivation is given in [M1]. □

The geodesic equation on $\mathcal{D}_\mu(M)$ is

$$\frac{\widetilde{\mathbf{D}}}{dt}\frac{d\eta}{dt} = 0, \qquad \eta(0) = \mathrm{id}, \quad \dot\eta(0) = X_o.$$

Using equation (2.9), the geodesic equation becomes, in terms of the Eulerian velocity field $X(t)$ defined by (2.8), the Euler equation of ideal incompressible flow:

$$\frac{\partial X}{\partial t} + \nabla_{X(t)}X(t) = -\nabla p(t), \qquad X(0) = X_o. \tag{2.11}$$

The pressure function $p(t)$ is written with a negative sign by convention, and comes from solving the equation (2.5):

$$\nabla p(t) = -Q\big(\nabla_{X(t)}X(t)\big). \tag{2.12}$$

The flow equation (2.8) can always be solved for η, with initial condition $\eta(0) = \mathrm{id}$, and this gives a one-to-one correspondence between solutions of (2.11) and geodesics starting at the identity. We will assume that $\eta(0) = \mathrm{id}$ from now on; by right-invariance, this is no loss of generality.

By formula (2.10), we may also think of the geodesic equation as a family of ordinary differential equations on the manifold:

$$\frac{D}{dt}\frac{d\eta}{dt}(t,x) = -\nabla p(t,x), \tag{2.13}$$

which is Newton's equation with a time-dependent potential $p(t,x)$. Of course, $p(t,x)$ is not given in advance, but determined by the fluid so as to preserve volume. This point of view will be useful later; we will see how one can consider the full linearized geodesic equation on $\mathcal{D}_\mu(M)$ by comparing it to the far simpler linearized Newton equation on M. Although these equations are not the same (when one is considering perturbations in M, one does not have the volume-preserving constraint to complicate the formulas), they are quite closely related.

We can eliminate the pressure term in (2.11) by computing the curl of both sides. In three dimensions, we get the vorticity form of the Euler equation:

$$\frac{\partial}{\partial t}\operatorname{curl} X(t,x) + \big[X(t,x), \operatorname{curl} X(t,x)\big] = 0. \tag{2.14}$$

Equation (2.14) implies that the vorticity is transported by the flow: for every $x \in M$,

$$\operatorname{curl} X\big(t, \eta(t,x)\big) = D\eta(t,x)\big(\operatorname{curl} X_o(x)\big). \tag{2.15}$$

More generally, by lowering indices in equation (2.11) to get an equation for the 1-form $X^\flat$ and taking the differential, we obtain the equation

$$\frac{\partial}{\partial t} dX^\flat + \mathcal{L}_X dX^\flat = 0, \tag{2.16}$$

the solution of which is

$$dX^\flat(t) = \big(\eta(t)^{-1}\big)^* dX_o^\flat. \tag{2.17}$$

See for example [AK] for details.

There are several formulas for the Riemann curvature tensor $\widetilde{\mathbf{R}}$ on $\mathcal{D}_\mu(M)$, but the only one we'll need is the following: if U, V, and W are divergence-free and tangent to the boundary, then

$$\widetilde{\mathbf{R}}(U,V)W = P\Big(R(U,V)W + \nabla_V Q(\nabla_U W) - \nabla_U Q(\nabla_V W)\Big). \tag{2.18}$$

See for example [P1]. Since the metric is right-invariant, the curvature tensor is as well, and thus we can compute the curvature at any $\eta \in \mathcal{D}_\mu(M)$ using the same formula.

We are interested in Jacobi fields, which are defined as follows: if $\eta(t,s)$ is a family of curves in $\mathcal{D}_\mu(M)$ with $\eta(t) = \eta(t,0)$ a geodesic, then $J(t) = \frac{\partial \eta}{\partial s}\big|_{s=0}$ is a Jacobi field along $\eta(t)$. Jacobi fields satisfy the linearized geodesic equation

$$\frac{\widetilde{\mathbf{D}}^2 J}{dt^2} + \widetilde{\mathbf{R}}\big(J(t), \dot\eta(t)\big)\dot\eta(t) = 0. \tag{2.19}$$

Equation (2.19) is extremely unwieldy, not least because the formulas for both curvature and the covariant derivative involve nonlocal operators (specifically, the solution of the Neumann problem (2.5)). However, the Jacobi equation can be simplified substantially, and in fact decoupled into two first-order equations. This fact was first observed by Rouchon [Ro], and exploited by the author [P1] to obtain explicit Jacobi fields along certain geodesics of $\mathcal{D}_\mu(M)$. These simplifications result from the following equivalent expressions for the linearization of the Newton equation (2.13).

Proposition 2.2. *Consider a solution $X(t)$ of the Euler equation* (2.11)*, with corresponding geodesic $\eta(t)$ in $\mathcal{D}_\mu(M)$. Let $J(t)$ be a vector field along $\eta(t)$. Then we have*

$$\frac{D^2 J}{dt^2} + \nabla_J \nabla p + R(J, \dot\eta)\dot\eta = \Big(\frac{\partial Z}{\partial t} + \nabla_X Z + \nabla_Z X\Big) \circ \eta, \tag{2.20}$$

where

$$Z = \frac{\partial Y}{\partial t} + [X, Y] \tag{2.21}$$

and $J = Y \circ \eta$.

We can also write

$$\frac{D^2 J}{dt^2} + \nabla_J \nabla p + R(J, \dot\eta)\dot\eta = (D\eta^{-1})^\star \left(\frac{\partial}{\partial t}\big(\Lambda V\big) + \big(\imath_V dX_o^\flat\big)^\sharp\right), \tag{2.22}$$

where $V = \partial_t U$ and $J = D\eta(U)$. Here $(D\eta^{-1})^\star$ is the pointwise metric adjoint of $D\eta^{-1}$, and $\Lambda = D\eta^\star D\eta$ is the metric pullback, a positive-definite linear operator on each T_xM. In addition, $(\imath_V dX_o^\flat)^\sharp$ is a vector field satisfying $\langle(\imath_V dX_o^\flat)^\sharp, F\rangle = dX_o^\flat(V, F) = \langle \nabla_V X_o, F\rangle - \langle \nabla_F X_o, V\rangle$ for any vector field F.

Proof. To obtain equation (2.20), we start with

$$\frac{DJ}{dt} = (\partial_t Y + \nabla_X Y)\circ\eta = (Z + \nabla_Y X)\circ\eta,$$

a consequence of (2.10). Using the Euler equation (2.11), we have

$$\begin{aligned}\Big(\frac{D^2 J}{dt^2} + \nabla_J\nabla p + R(J,\dot\eta)\dot\eta\Big)\circ\eta^{-1} &= \frac{D}{dt}\big((Z + \nabla_Y X)\circ\eta\big)\circ\eta^{-1} + \nabla_Y\nabla p + R(Y,X)X \\ &= \partial_t(Z + \nabla_Y X) + \nabla_X(Z + \nabla_Y X) - \nabla_Y(\partial_t X + \nabla_X X) + R(Y,X)X \\ &= \partial_t Z + \nabla_{\partial_t Y} X + \nabla_Y(\partial_t X) + \nabla_X Z + \nabla_X\nabla_Y X - \nabla_Y(\partial_t X) - \nabla_Y\nabla_X X \\ &\qquad + \nabla_Y\nabla_X X - \nabla_X\nabla_Y X + \nabla_{[X,Y]}X \\ &= \partial_t Z + \nabla_Z X + \nabla_X Z.\end{aligned}$$

To derive equation (2.22), we first recall that $Y = \eta_* U$. By the definition of the Lie bracket (see for example Spivak [Sp]), we have

$$\eta_* \frac{\partial}{\partial t} \eta_*^{-1} Y = \frac{\partial Y}{\partial t} + [X, Y]. \tag{2.23}$$

Thus $Z = \eta_* V$. For convenience, define $L = Z \circ \eta = D\eta(V)$.

Then by equations (2.20) and (2.10), our goal becomes to prove that

$$\frac{DL}{dt} + \nabla_L X = (D\eta^{-1})^\star \left(\frac{\partial}{\partial t} \big(\Lambda V\big) + \big(\imath_V dX_o^\flat\big)^\sharp \right). \tag{2.24}$$

This equation involves no space derivatives, so we can consider it as an equation along the fixed curve $\eta(t, x)$ for each particular $x \in M$.

So for some fixed x, pick an arbitrary vector $w_o \in T_x M$. Then we can compute

$$\Big\langle w_o, \frac{d}{dt}(\Lambda V) \Big\rangle = \frac{d}{dt} \langle w_o, \Lambda V \rangle = \frac{d}{dt} \langle D\eta(w_o), L \rangle. \tag{2.25}$$

By equations (2.23) and (2.10), we can compute that

$$\frac{D}{dt}\big(D\eta(w_o)\big) = \nabla_{D\eta(w_o)} X.$$

Thus (2.25) yields

$$\begin{aligned} \Big\langle w_o, \frac{d}{dt}(\Lambda V) \Big\rangle &= \big\langle \nabla_{D\eta(w_o)} X, L \big\rangle + \Big\langle D\eta(w_o), \frac{DL}{dt} \Big\rangle \\ &= \big\langle \nabla_{D\eta(w_o)} X, L \big\rangle - \big\langle D\eta(w_o), \nabla_L X \big\rangle + \Big\langle D\eta(w_o), \frac{DL}{dt} + \nabla_L X \Big\rangle. \end{aligned}$$

Using the general formula

$$\langle \nabla_A X, B \rangle - \langle \nabla_B X, A \rangle = dX^\flat(A, B),$$

we can write

$$\Big\langle w_o, \frac{d}{dt}(\Lambda V) \Big\rangle = \Big\langle D\eta(w_o), \frac{DL}{dt} + \nabla_L X \Big\rangle - dX^\flat\big(D\eta(V), D\eta(w_o)\big).$$

Now since the vorticity 2-form is transported by the flow, equation (2.17) yields

$$dX^\flat\big(D\eta(V), D\eta(w_o)\big) = dX_o^\flat(V, w_o) = \langle \big(\imath_V dX_o^\flat\big)^\sharp, w_o \rangle.$$

Thus we finally get

$$\Big\langle w_o, \frac{d}{dt}(\Lambda V) \Big\rangle = \Big\langle D\eta(w_o), \frac{DL}{dt} + \nabla_L X \Big\rangle - \langle \big(\imath_V dX_o^\flat\big)^\sharp, w_o \rangle,$$

and since this is true for any $w_o \in T_x M$, we have the equation

$$D\eta^\star \Big(\frac{DL}{dt} + \nabla_L X \Big) = \frac{d}{dt}(\Lambda V) + \big(\imath_V dX_o^\flat\big)^\sharp,$$

which yields (2.24) and hence (2.22). □

The following proposition shows how the Jacobi equation simplifies under either left- or right-translations.

Proposition 2.3. *If η is a geodesic in $\mathcal{D}_\mu(M)$ and X is its Eulerian velocity field defined by* (2.8)*, then the Jacobi operator in* (2.19) *can be written in two ways:*

- *in terms of the right-translation, with $Y = dR_{\eta^{-1}}(J)$, as*

$$\frac{\widetilde{\mathbf{D}}^2 J}{dt^2} + \widetilde{\mathbf{R}}\big(J, \dot\eta\big)\dot\eta = dR_\eta \left(\frac{\partial Z}{\partial t} + P\Big(\nabla_Z X + \nabla_X Z \Big) \right), \tag{2.26}$$

  where

$$Z = \partial_t Y + [X, Y].$$

- *in terms of the left-translation, with $U = dL_{\eta^{-1}}\big(J\big)$, as*

$$\frac{\widetilde{\mathbf{D}}^2 J}{dt^2} + \widetilde{\mathbf{R}}\big(J, \dot\eta\big)\dot\eta = (dL_{\eta^{-1}})^\star \left(\frac{\partial}{\partial t} P\Big(\Lambda \frac{\partial U}{\partial t} \Big) + K_{X_o}\Big(\frac{\partial U}{\partial t} \Big) \right), \tag{2.27}$$

  where the operator $K_{X_o}\colon T_{id}\mathcal{D}_\mu(M) \to T_{id}\mathcal{D}_\mu(M)$ is defined by

$$K_{X_o}(W) = P\big(\imath_W dX_o^\flat \big)^\sharp, \tag{2.28}$$

  with $\Lambda = D\eta^\star D\eta$ being the metric pullback, X_o being the initial velocity field, and

$$(dL_{\eta^{-1}})^\star = dR_\eta \circ P \circ (D\eta^{-1})^\star \circ dR_{\eta^{-1}}$$

  being the L^2 adjoint of the operator $dL_{\eta^{-1}}\colon T_{id}\mathcal{D}_\mu \to T_{\eta^{-1}}\mathcal{D}_\mu$.

Proof. The right-translated Jacobi equation (2.26) was derived by Rouchon [Ro], by linearizing the geodesic equation (2.11) and (2.8) directly. The fact that (2.26) is equivalent to (2.19) can be seen directly, using formulas (2.9) and (2.18).

Equation (2.27) is a consequence of (2.22), along with the observation that for any vector field W, we have $(dL_{\eta^{-1}})^\star(W) = (dL_{\eta^{-1}})^\star \circ P(W)$. This observation follows from the fact that if g is any function on M, then

$$(dL_{\eta^{-1}})^\star(\nabla g) = dR_\eta \circ P \circ (D\eta^{-1})^\star \circ dR_{\eta^{-1}}(\nabla g) = dR_\eta \circ P\big(\nabla (g \circ \eta^{-1}) \big) = 0,$$

so that $(dL_{\eta^{-1}})^\star \circ Q \equiv 0$. □

The operator K_{X_o} defined by (2.28) is given in two dimensions by

$$K_{X_o}(W) = P\big((\operatorname{curl} X_o) \star W \big),$$

where $\star$ is the two-dimensional Hodge star operator that rotates vectors 90°. This operator is compact, as discussed in [EMP]. In three dimensions,

$$K_{X_o}(W) = P\big(\operatorname{curl} X_o \times W \big),$$

and this operator is generally not compact. The fact that this operator fails to be compact in three dimensions is the main reason Fredholmness of the exponential map fails in three dimensions, which is why conjugate points look so different between $\mathcal{D}_\mu(M^2)$ and $\mathcal{D}_\mu(M^3)$.

The main thing we are interested in for this paper is the index form along a geodesic $\eta(t)$ in $\mathcal{D}_\mu(M)$. In general this is defined for a Riemannian manifold as

$$I_a\big(J(t),J(t)\big) = \int_0^a \Big\langle\Big\langle \frac{\widetilde{\mathbf{D}}J}{dt}, \frac{\widetilde{\mathbf{D}}J}{dt}\Big\rangle\Big\rangle - \Big\langle\Big\langle \widetilde{\mathbf{R}}\Big(J(t), \frac{d\eta}{dt}\Big)\frac{d\eta}{dt}, J(t)\Big\rangle\Big\rangle \, dt. \tag{2.29}$$

The index form represents the second derivative of the energy functional

$$E(s) = \frac{1}{2}\int_0^a \Big\langle\Big\langle \frac{\partial\eta(t,s)}{\partial t}, \frac{\partial\eta(t,s)}{\partial t}\Big\rangle\Big\rangle \, dt.$$

If $\eta(t,s)$ is a family of curves in $\mathcal{D}_\mu(M)$, such that $\eta(t,0)$ is a geodesic, with $\eta(0,s)$ and $\eta(a,s)$ constant in s, then $E'(0)=0$ and

$$E''(0) = I_a\Big(\frac{\partial\eta}{\partial s}\Big|_{s=0}, \frac{\partial\eta}{\partial s}\Big|_{s=0}\Big).$$

So if the index form is negative for some vector field $J(t)$ vanishing at $t=0$ and $t=a$, then the geodesic is not minimizing on $[0,a]$. In addition, there must be a Jacobi field which vanishes at $t=0$ and $t=b$ for some $b\in(0,a)$.

We will derive an alternative formula for the index form (2.29), which will form the basis for the rest of the paper.

Proposition 2.4. *If $\eta(t)$ is a geodesic in $\mathcal{D}_\mu(M)$ and $J(t)$ is a smooth vector field along $\eta(t)$ vanishing at $t=0$ and $t=a$, then the index form $I_a\big(J(t),J(t)\big)$ may be written in terms of the left-translation $U(t) = dL_{\eta(t)^{-1}}\big(J(t)\big)$ as*

$$\begin{aligned} I_a\big(J(t),J(t)\big) = \int_0^a \int_M &\Big\langle \Lambda(t,x)\frac{\partial U}{\partial t}(t,x), \frac{\partial U}{\partial t}(t,x)\Big\rangle \\ &+ dX_o(x)^\flat\Big(U(t,x), \frac{\partial U}{\partial t}(t,x)\Big)\, \mu(x)\, dt, \end{aligned} \tag{2.30}$$

where $\Lambda(t,x) = D\eta(t,x)^\star D\eta(t,x)$ is the metric pullback and $dX_o^\flat$ is the initial vorticity 2-form.

Proof. The formula follows immediately from Proposition 2.3, after integrating the index form (2.29) by parts to obtain

$$I_a\big(J(t),J(t)\big) = -\int_0^a \Big\langle\Big\langle \frac{\widetilde{\mathbf{D}}^2 J}{dt} + \widetilde{\mathbf{R}}\Big(J(t), \frac{d\eta}{dt}\Big)\frac{d\eta}{dt}, J(t)\Big\rangle\Big\rangle \, dt.$$

□

What is remarkable about the formula (2.30) is that it involves only local computations; it is not necessary to solve the Neumann problem (2.5) to compute the index form. The index form is virtually the *only* object in the geometry of $\mathcal{D}_\mu$ that can be computed so easily, and it is this fact which helps so much to understand conjugate points on $\mathcal{D}_\mu$, despite our very incomplete understanding of the curvature on $\mathcal{D}_\mu$.

The main reason we use left translations to write the index form (2.30) is that it yields an index form in each tangent space T_xM, so that we can study the differential equation in a single vector space rather than along a Lagrangian path. However, the two approaches are equivalent.

3. The local criterion

Theorem 3.1. *Let M be a 3-dimensional compact manifold (possibly with boundary). Let $\eta\colon [0,T) \to \mathcal{D}_\mu(M)$ be a geodesic curve in the diffeomorphism group with $\eta(0) = id$. (Here T is the maximal time of existence, which may be infinite.) Let $X(t) = \frac{d\eta}{dt} \circ \eta(t)^{-1}$ be the velocity field, with $X(0) = X_o$.*

If for some point x in the interior of M, the ordinary differential equation

$$\frac{d}{dt}\left(\Lambda(t,x)\,\frac{du}{dt}\right) + \operatorname{curl} X_o(x) \times \frac{du}{dt} = 0 \tag{3.31}$$

has a nontrivial solution vanishing at $t = 0$ and $t = a$, then for any $\delta > 0$, there is a $b \in (0, a+\delta)$ such that $\eta(b)$ is monoconjugate to id along η.

Proof. Clearly $\operatorname{curl} X_o(x)$ is not zero; if it were, we could not have a nontrivial solution vanishing at two points, since $\Lambda(t,x)$ is positive definite.

So set up an oriented orthonormal basis $\{e_1, e_2, e_3\}$ at T_xM, such that $\operatorname{curl} X_o(x) = \omega_o e_3$, with $\omega_o > 0$. Choose Riemannian normal coordinates (x_1, x_2, x_3) in a neighborhood of x interior to M, such that at x, $\partial_{x_1} = e_1$, $\partial_{x_2} = e_2$, and $\partial_{x_3} = e_3$. Let $h\colon \mathbb{R} \to \mathbb{R}$ be a nontrivial C^∞ function which vanishes identically outside $[-1,1]$.

For a small $\varepsilon > 0$, let us define three vector fields

$$\begin{aligned}
A_1 &= \varepsilon^4 h'\Big(\frac{x_1}{\varepsilon}\Big) h\Big(\frac{x_2}{\varepsilon^2}\Big) h\Big(\frac{x_3}{\varepsilon^3}\Big)\,\partial_{x_1} - \varepsilon^3 h\Big(\frac{x_1}{\varepsilon}\Big) h'\Big(\frac{x_2}{\varepsilon^2}\Big) h\Big(\frac{x_3}{\varepsilon^3}\Big)\,\partial_{x_2},\\
A_2 &= \varepsilon^3 h\Big(\frac{x_1}{\varepsilon}\Big) h'\Big(\frac{x_2}{\varepsilon^2}\Big) h\Big(\frac{x_3}{\varepsilon^3}\Big)\,\partial_{x_1} + \varepsilon^4 h'\Big(\frac{x_1}{\varepsilon}\Big) h\Big(\frac{x_2}{\varepsilon^2}\Big) h\Big(\frac{x_3}{\varepsilon^3}\Big)\,\partial_{x_2},\\
A_3 &= -\frac{\varepsilon^3}{2} h'\Big(\frac{x_1}{\varepsilon^2}\Big) h\Big(\frac{x_2}{\varepsilon^3}\Big) h\Big(\frac{x_3}{\varepsilon}\Big)\,\partial_{x_1} + \frac{\varepsilon^2}{2} h\Big(\frac{x_1}{\varepsilon^2}\Big) h'\Big(\frac{x_2}{\varepsilon^3}\Big) h\Big(\frac{x_3}{\varepsilon}\Big)\,\partial_{x_2}.
\end{aligned}$$

(We set each $A_j \equiv 0$ outside the coordinate neighborhood.)

Now we specify divergence-free vector fields E_1, E_2, and E_3 by the formulas $E_j = \operatorname{curl} A_j$. Since we are working in Riemannian normal coordinates, we can compute these curls to order $O(\varepsilon)$ just using the Euclidean formulas, and we obtain:

$$\begin{aligned}
E_1 &= h\Big(\frac{x_1}{\varepsilon}\Big) h'\Big(\frac{x_2}{\varepsilon^2}\Big) h'\Big(\frac{x_3}{\varepsilon^3}\Big)\,\partial_{x_1} + O(\varepsilon) \text{ on } [-\varepsilon,\varepsilon]\times[-\varepsilon^2,\varepsilon^2]\times[-\varepsilon^3,\varepsilon^3],\\
E_2 &= h\Big(\frac{x_1}{\varepsilon}\Big) h'\Big(\frac{x_2}{\varepsilon^2}\Big) h'\Big(\frac{x_3}{\varepsilon^3}\Big)\,\partial_{x_2} + O(\varepsilon) \text{ on } [-\varepsilon,\varepsilon]\times[-\varepsilon^2,\varepsilon^2]\times[-\varepsilon^3,\varepsilon^3],\\
E_3 &= h'\Big(\frac{x_1}{\varepsilon^2}\Big) h'\Big(\frac{x_2}{\varepsilon^3}\Big) h\Big(\frac{x_3}{\varepsilon}\Big)\,\partial_{x_3} + O(\varepsilon) \text{ on } [-\varepsilon^2,\varepsilon^2]\times[-\varepsilon^3,\varepsilon^3]\times[-\varepsilon,\varepsilon].
\end{aligned}$$

These vector fields are chosen so that, roughly speaking, E_i is nearly parallel to e_i near x, to lowest order. More precisely, we can check that the following formulas hold for the L^2 inner products:

$$\langle\langle E_1, E_1\rangle\rangle = \langle\langle E_2, E_2\rangle\rangle = \langle\langle E_3, E_3\rangle\rangle = \Gamma\varepsilon^6 + O(\varepsilon^7)$$
$$\langle\langle E_1, E_2\rangle\rangle = \langle\langle E_1, E_3\rangle\rangle = \langle\langle E_2, E_3\rangle\rangle = O(\varepsilon^7)$$
$$\langle\langle \partial_{x_3} \times E_1, E_2\rangle\rangle = \Gamma\varepsilon^6 + O(\varepsilon^7), \qquad \langle\langle \partial_{x_3} \times E_2, E_1\rangle\rangle = -\Gamma\varepsilon^6 + O(\varepsilon^7)$$
$$\langle\langle \partial_{x_3} \times E_1, E_3\rangle\rangle = \langle\langle \partial_{x_3} \times E_2, E_3\rangle\rangle = O(\varepsilon^7),$$

where the constant Γ is defined by

$$\Gamma = \left(\int_{-1}^{1} h(\sigma)^2\, d\sigma\right)\left(\int_{-1}^{1} h'(\sigma)^2\, d\sigma\right)^2.$$

Now since $u(t)$ is a solution of equation (3.31) vanishing at $t = 0$ and $t = a$, we know that there is a vector function $\tilde{u}(t)$ vanishing at $t = 0$ and $t = a + \delta$ such that

$$i_{a+\delta}(\tilde{u}, \tilde{u}) \equiv \int_0^{a+\delta} \langle \Lambda(t, x)\partial_t \tilde{u}(t), \partial_t \tilde{u}(t)\rangle + \langle \operatorname{curl} X_o(x) \times \tilde{u}(t), \partial_t \tilde{u}(t)\rangle\, dt < 0.$$

(The construction is the same as that for Jacobi fields in finite-dimensional Riemannian geometry, or more generally for index forms of second-order self-adjoint equations. See for example Reid [Re].)

If $\tilde{u}(t) = u^1(t)e_1 + u^2(t)e_2 + u^3(t)e_3$, then define $\tilde{U}(t) = u^1(t)E_1 + u^2(t)E_2 + u^3(t)E_3$. For y in the support of $\tilde{U}$, we can approximate $\Lambda(t, y) = \Lambda(t, x) + O(\varepsilon)$ and $\operatorname{curl} X_o(y) = \operatorname{curl} X_o(x) + O(\varepsilon)$. Therefore, we have

$$\begin{aligned} I_{a+\delta}(\tilde{U}, \tilde{U}) &= \int_0^{a+\delta} \langle\langle \Lambda(t)\partial_t \tilde{U}, \partial_t \tilde{U}\rangle\rangle + \langle\langle \operatorname{curl} X_o \times \tilde{U}(t), \partial_t \tilde{U}\rangle\rangle\, dt \\ &= \int_0^{a+\delta} \langle\langle \Lambda(t)\partial_t \tilde{U}, \partial_t \tilde{U}\rangle\rangle + \omega_o \langle\langle \partial_{x_3} \times \tilde{U}, \partial_t \tilde{U}\rangle\rangle\, dt + O(\varepsilon^7) \\ &= \Gamma\varepsilon^6 i_{a+\delta}(\tilde{u}, \tilde{u}) + O(\varepsilon^7), \end{aligned}$$

and choosing ε sufficiently small, we can make this quantity negative.

Since the index form is negative for some divergence free vector field on the interval $[0, a + \delta]$, there must be a Jacobi field along η vanishing at $t = 0$ and $t = b$ for some $b < a + \delta$. So $\eta(b)$ is monoconjugate to $\eta(0)$ along η, as desired. □

Remark 3.2. The main point is that for any particular vector $u \in T_xM$, we can construct a divergence-free vector field U such that $U \approx u$ near x and $P(\operatorname{curl} X_o \times U) \approx \operatorname{curl} X_o \times u$ near x. We can do this only in three (or possibly higher) dimensions.

In two dimensions, the index form takes the form

$$I_a(U, U) = \int_0^a \langle\langle \Lambda(t)\partial_t U, \partial_t U\rangle\rangle + \langle\langle (\operatorname{curl} X_o) \star U, \partial_t U\rangle\rangle\, dt,$$

where $\star$ is the Hodge star operator. If U is any divergence-free vector field with support in a disc, then $\star U$ is a gradient, and thus to lowest order, $(\operatorname{curl} X_o) \star U$ is also a gradient. Since the gradients are orthogonal to the divergence-free vector fields, the second term in the index form vanishes to lowest order; thus the index form is positive definite to lowest order.

We conclude that there is no local criterion that can be used to find conjugate points along two-dimensional fluid flows: conjugate points on $\mathcal{D}_\mu(M^2)$ are an essentially global phenomenon. In three (and possibly higher) dimensions, conjugate points are essentially a local phenomenon.

Remark 3.3. The result is sharp, in the sense that there may not be a monoconjugate point actually at $\eta(a)$. This is precisely what happens for one example where we can compute everything explicitly: uniform rotation with angular velocity 1 of the solid torus $D^2 \times S^1$. Ebin, Misiołek, and the author [EMP] computed explicitly the Jacobi fields along this flow in terms of curl eigenfields on the cylinder, and found that monoconjugate points occur at a sequence of locations that decreases to π, but that $\eta(\pi)$ itself is not a monoconjugate point. In this example $\Lambda(t,x)$ is always the identity and $\operatorname{curl} X_o \equiv 2\,\partial_z$, so that the equation (3.31) becomes $u''(t) + 2\,\partial_z \times u'(t) = 0$. With $u(0) = 0$, the solutions are

$$u(t) = \begin{pmatrix} \sin^2 t & -\sin t\cos t & 0 \\ \sin t \cos t & \sin^2 t & 0 \\ 0 & 0 & t \end{pmatrix} u'(0),$$

and choosing $u'(0)$ orthogonal to ∂_z, we see that the first vanishing point is $a = \pi$.

If a is not actually a monoconjugate location, then there must be a sequence of monoconjugate locations descending to a. We will discuss this point more thoroughly in the next section.

Remark 3.4. By Proposition 2.2, the equation (3.31) is equivalent to the pair of equations

$$\frac{Dy}{dt} - \nabla_{y(t)} X = z(t) \quad \text{and} \quad \frac{Dz}{dt} + \nabla_{z(t)} X = 0 \tag{3.32}$$

for vector fields $y(t)$ and $z(t)$ along a Lagrangian path $t \mapsto \eta(t)(x)$ in M^3. For a steady flow X for which we happen to know a Lagrangian path, equations (3.32) will often be easier to write down and solve than equation (3.31). (Of course, at a fixed point of a steady flow, the two approaches are the same.)

We can also write equations (3.32) as the single second-order equation

$$\frac{D^2 y}{dt^2} + \nabla_y \nabla p + R(y, \dot\eta)\dot\eta = 0. \tag{3.33}$$

This is the linearization of the Newton equation $\frac{D}{dt}\dot\eta = -\nabla p$ on the manifold. Thus if the finite-dimensional Newtonian system (for the time-dependent potential energy function p) has a conjugate point, so does the infinite-dimensional Riemannian system. One can use the same sort of comparison techniques to find solutions of (3.33) as are used in finite-dimensional Riemannian geometry.

Theorem 3.1 is easiest to apply if $X = X_o$ is a steady solution of the Euler equation, with a fixed point x.

Proposition 3.5. *Suppose X is a steady solution of the Euler equation $\nabla_X X = -\nabla p$ on a 3-manifold M, and x is a fixed point of X in the interior of M. If $\Delta p(x) > 0$, then equation* (3.31) *has nontrivial solutions vanishing at both $t = 0$ and $t = \pi\sqrt{2/\Delta p(x)}$. Otherwise, all solutions of* (3.31) *can vanish at most at one time.*

Proof. The operator $u \mapsto \nabla_u X$ is a linear operator in $T_x M$. Since X is assumed to be a steady solution of the Euler equation, we know by (2.14) that $[X, \operatorname{curl} X] = 0$ everywhere, and in particular at x. Thus $\nabla_{\operatorname{curl} X(x)} X = \nabla_{X(x)} \operatorname{curl} X = 0$, because $X(x) = 0$. In addition, for any $u \in T_x M$, we have $\langle \nabla_u X, \operatorname{curl} X(x)\rangle - \langle \nabla_{\operatorname{curl} X(x)} X, u\rangle = \langle \operatorname{curl} X(x) \times \operatorname{curl} X(x), u\rangle = 0$. Therefore, in a basis $\{e_1, e_2, e_3\}$ with e_3 parallel to $\operatorname{curl} X(x)$, the matrix of $u \mapsto \nabla_u X$ is of the form

$$\nabla_u X = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} u,$$

where A is a 2×2 matrix. Since $\operatorname{div} X = 0$, we have $\operatorname{Tr} A = 0$. So the characteristic equation for A is $A^2 = -(\det A) I$.

We have the general formula $\operatorname{div} \nabla_X X = \operatorname{Ric}(X, X) + \operatorname{Tr}(u \mapsto \nabla_{\nabla_u X} X)$, which is valid for any divergence-free vector field. Thus for a solution of equation (2.11), we have $\Delta p + \operatorname{Ric}(X, X) = -\operatorname{Tr}(u \mapsto \nabla_{\nabla_u X} X)$. In particular, at the point x, we have $\operatorname{Ric}(X(x), X(x) = 0$ and thus

$$\Delta p(x) = -\operatorname{Tr}(u \mapsto \nabla_{\nabla_u X} X) = -\operatorname{Tr} A^2 = 2 \det A.$$

The solution of the pair of equations

$$\frac{dz}{dt} + \nabla_z X = 0, \qquad \frac{dy}{dt} - \nabla_y X = z$$

with initial conditions $y(0) = 0$ and $z(0) = z_o$ is

$$y(t) = \begin{pmatrix} \frac{1}{2} A^{-1}(e^{tA} - e^{-tA}) & 0 \\ 0 & t \end{pmatrix} z_o.$$

If $\Delta p(x) > 0$, then $e^{tA} = \cos\left(\sqrt{\det A}\, t\right) I + \frac{1}{\sqrt{\det A}} \sin\left(\sqrt{\det A}\, t\right) A$, so that

$$y(t) = \begin{pmatrix} \frac{1}{\sqrt{\det A}} \sin\left(\sqrt{\det A}\, t\right) I & 0 \\ 0 & t \end{pmatrix} z_o.$$

Choosing z_o orthogonal to e_3, we obtain a nontrivial solution which vanishes at time $t = \pi\sqrt{2/\Delta p(x)}$. On the other hand, if $\Delta p(x) \le 0$, we can easily verify that each component of $y(t)$ increases with time, so that there are no nontrivial solutions vanishing at two times. □

We have a natural converse to Theorem 3.1, which works for any dimension $n \ge 2$.

Proposition 3.6. *Let M be any manifold with dimension $n \ge 2$, possibly with boundary. Suppose $\eta(t)$ is a geodesic in $\mathcal{D}_\mu(M)$ with $\eta(0) = id$, and let X be the Eulerian velocity field defined by $X = \frac{\partial \eta}{\partial t} \circ \eta^{-1}$.*

If there is a Jacobi field along η vanishing at $t = 0$ and $t = a > 0$, then for some x in the interior of M, there is a solution $u(t)$ of the ordinary differential equation

$$\frac{d}{dt}\left(\Lambda(t,x)\,\frac{du}{dt} + \big(\imath_{u(t)} dX_o^\flat(x)\big)^\sharp\right) = 0 \tag{3.34}$$

with $u(0) = u(\alpha) = 0$ for some $\alpha \in (0, a]$.

Proof. Let $U(t)$ be the left translation of the Jacobi field vanishing at $t = 0$ and $t = a$. Then the index form $I_a(U, U)$ vanishes:

$$I_a(U,U) = \int_0^a \langle\langle \Lambda(t)U_t(t), U_t(t)\rangle\rangle + \langle\langle \big(\imath_{U(t)} dX_o^\flat\big)^\sharp, U_t(t)\rangle\rangle\, dt = 0.$$

Thus, interchanging the order of integration, we know that

$$\int_M \int_0^a \big\langle \Lambda(t,x)U_t(t,x), U_t(t,x)\big\rangle + \big\langle \big(\imath_{U(t,x)} dX_o^\flat(x)\big)^\sharp, U_t(t,x)\big\rangle\, dt\, d\mu(x) = 0.$$

As a result, we know

$$I_a(U,U) = \int_M i_a(x)\, d\mu(x) = 0,$$

where the integrand is

$$i_a(x) = \int_0^a \langle \Lambda(t,x)U_t(t,x), U_t(t,x)\rangle + \big\langle \big(\imath_{U(t,x)} dX_o^\flat(x)\big)^\sharp, U_t(t,x)\big\rangle\, dt. \tag{3.35}$$

Thus $i_a(x)$ must vanish for some x in the interior of M.

Now $i_a(x)$ is the index form of the self-adjoint system (3.34), and since the matrix $\Lambda(t,x)$ is always positive-definite, we can apply the Morse index theorem for systems to conclude that if $i_a(x) = 0$, then there is some solution of (3.34) vanishing at $t = 0$ and at some $t = \alpha \le a$. See Reid [Re], Theorem V.8.1. □

Remark 3.7. As in Remark 3.4, we can also use the equations (3.32), or the equivalent (3.33), instead of (3.34).

Example 3.8. Consider M^2, a two-dimensional disc, sphere, annulus, or torus, with rotationally symmetric metric of the form

$$ds^2 = dr^2 + \varphi^2(r)\, d\theta^2 \tag{3.36}$$

and a vector field

$$X = u(r)\,\partial_\theta. \tag{3.37}$$

Any such X is a steady solution of the Euler equation (2.11), and thus generates a geodesic η given in coordinates by $\eta(t)\big(r,\theta\big) = \big(r, \theta + tu(r)\big)$. These are the simplest nontrivial steady Euler flows.

In [P2], the author proved the following theorem.

Theorem 3.9. *A geodesic in $\mathcal{D}_\mu(M^2)$ generated by an analytic steady flow X on M^2 with isolated singularities has nonpositive curvature operator all along it if and only if M^2 is a disc, sphere, annulus, or torus with a polar coordinate system with metric* (3.36) *in which X has the form* (3.37)*, and in addition:*

- *if M^2 is a torus, φ is constant;*
- *if M^2 is not a torus, then the function $Q(r) = (\varphi' u)'/u'$ is defined for all r and satisfies the differential inequality*

$$\varphi Q' + Q^2 \le 1 \tag{3.38}$$

everywhere.

Nonpositive curvature operator implies, by the Rauch comparison theorem, that the differential of the exponential map satisfies $|d(\exp_{\mathrm{id}})_{tX}(V)| \ge t|V|$ for any t and any $V \in T_{\mathrm{id}}\mathcal{D}_\mu(M^2)$; therefore there are no conjugate points (monoconjugate or epiconjugate—see Grossman [G]).

Proposition 3.6 gives us a simpler criterion for the nonexistence of monoconjugate points for flows of the form $X = u(r)\,\partial_\theta$. We can prove the following.

Proposition 3.10. *Suppose M^2 is a disc, sphere, annulus, or torus, with metric given in polar coordinates by* (3.36)*, with a vector field X of the form* (3.37)*. Define a radial function by*

$$A(r) = 4u(r)^2\varphi'(r)^2 + 2\varphi(r)\varphi'(r)u(r)u'(r). \tag{3.39}$$

If $A(r) \le 0$ for all r, then the geodesic η in $\mathcal{D}_\mu(M^2)$ defined by X has no monoconjugate points. Furthermore, if $A(r) > 0$ for some r, then the first monoconjugate point along η (if there is one) cannot occur earlier than $\tau = 2\pi/\sqrt{\sup_r A(r)}$.

Proof. Using Remark 3.7, if $\eta(a)$ is monoconjugate to id along η, then along any circle of constant r, the equations (3.32) have a solution with $y(0) = 0$ and $y(a) = 0$.

With $y(t) = f(t)\,\partial_r + g(t)\,\partial_\theta$ and $z(t) = h(t)\,\partial_r + k(t)\,\partial_\theta$, equations (3.32) take the form

$$\dot f(t) = h(t) \qquad \dot g(t) = u'(r) f(t) + k(t)$$

$$\dot h(t) = 2u(r)\varphi(r)\varphi'(r)k(t) \qquad \dot k(t) = -\left(u'(r) + 2\frac{\varphi'(r)u(r)}{\varphi(r)}\right) h(t).$$

The equation for $h(t)$ takes the form

$$\ddot h(t) + A(r)h(t) = 0,$$

and from here we can easily solve to find h, k, f, and g with initial conditions $h(0) = h_o$, $k(0) = k_o$, $f(0) = 0$, and $g(0) = 0$. It is not hard to see that these equations have solutions vanishing for $t > 0$ if and only if we have $A(r) > 0$, and then the solution vanishes at $2\pi/\sqrt{A(r)}$.

Thus if $A(r) \le 0$ for all r, the equation (3.32) does not have a solution vanishing at two times along any Lagrangian path, and thus the geodesic cannot have any monoconjugate pairs. On the other hand, if $\eta(0)$ and $\eta(a)$ are monoconjugate, then for some

r, we must have $2\pi/\sqrt{A(r)} = a$. As a result, the first possible monoconjugate point cannot occur before $2\pi/\sqrt{\sup_r A(r)}$. □

Theorem 3.9 and Proposition 3.10 do actually give us distinct criteria for a two-dimensional rotational flow to have no monoconjugate points (and thus to be infinitesimally minimizing along its entire length). For example, if $\varphi(r) = r$ and $u(r) \equiv 1$ on the disc, then Theorem 3.9 implies that the geodesic has nonpositive curvature operator and thus no conjugate points, while Proposition 3.10 is inconclusive.

On the other hand, if $\varphi(r) = \sin r$ and $u(r) = 1/\sin^2(r)$ on the portion $\pi/4 < r < 3\pi/4$ of the round sphere, then $A(r) \equiv 0$, so that the geodesic has no monoconjugate points; however, $Q(r) = (1 + \cos^2 r)/(2\cos r)$ is singular at $r = \pi/2$ and never satisfies the differential inequality (3.38). Thus the curvature operator along the geodesic is sometimes positive and sometimes negative.

4. The first conjugate point

Proposition 4.1. *Suppose η is a geodesic in $\mathcal{D}_\mu(M^3)$. Let*

$$\tau = \inf\{a > 0 \,\big|\, \eta(a) \textit{ is monoconjugate to } \eta(0) \textit{ along } \eta\}.$$

For each x in the interior of M^3, let

$$\tau(x) = \inf\{a > 0 \,\big|\, \textit{some solution of equation } (3.31) \textit{ vanishes at } t = 0 \textit{ and } t = a.\}$$

Then $\tau = \inf_{x\in int(M)} \tau(x)$. If $\eta(\tau)$ is itself monoconjugate to $\eta(0)$ along η, then in addition $\tau(x)$ is constant on M.

Proof. Theorem 3.1 implies that $\tau \le \inf_{x\in \mathrm{int}(M)} \tau(x)$, while Proposition 3.6 implies that $\inf_{x\in\mathrm{int}(M)} \tau(x) \le \tau$. This proves the first claim.

Now suppose $\eta(\tau)$ is actually monoconjugate to $\eta(0)$. Then there is a Jacobi field $J(t)$ along η with $J(0) = 0$ and $J(\tau) = 0$. As in the proof of Proposition 3.6, the left-translation $U(t)$ of this vector field satisfies the equation

$$\int_M i_\tau(x)\, d\mu(x) = 0,$$

where the integrand is

$$i_\tau(x) = \int_0^\tau \langle \Lambda(t,x) U_t(t,x), U_t(t,x)\rangle + \langle \big(\imath_{U(t,x)} dX_o^\flat(x)\big)^\sharp, U_t(t,x)\rangle\, dt.$$

If $i_\tau(x) < 0$ at some point x, then since $i_\tau(x)$ is the index form of the self-adjoint system (3.31) in T_xM, there must be a solution of (3.31) vanishing at $t = 0$ and $t = a$ for some $a < \tau$. (As before, see Reid [Re].) This implies $\tau(x) \le a < \tau$, which is impossible. Thus $i_\tau(x) \ge 0$ for every x in the interior of M. The only way a nonnegative function can integrate to zero is if it identically vanishes, so $i_\tau(x) = 0$ for every x. Thus we must have $\tau(x) = \tau$ for every x. □

If we know anything about the metric pullback Λ, the equation (3.31) will be easy to solve, and we can determine the exact location of the first conjugate point. The easiest case is of course when X is a Killing field.

Corollary 4.2. *If the geodesic $\eta(t)$ in $\mathcal{D}_\mu(M^3)$ consists of isometries, then the Eulerian velocity field X is steady and a Killing field. The infimum of monoconjugate point locations is then*

$$\tau = \frac{2\pi}{\sup_M |\operatorname{curl} X|}.$$

If $\eta(\tau)$ is itself monoconjugate to id, then $\operatorname{curl} X$ has constant length on M.

Proof. We know that $\eta'(0) = X_o$ is a Killing field by definition, and since any Killing field is a steady solution of the Euler equation (see Misiołek [M1] for the proof), we must have $X(t) = X_o$ for all t.

Since $\eta(t)$ is always an isometry, the deformation $\Lambda(t, x)$ is always the identity, so that equation (3.31) becomes (using $\omega = \operatorname{curl} X$)

$$\frac{d^2u}{dt^2} + \omega(x) \times \frac{du}{dt} = 0,$$

whose solution with $u(0) = 0$ and $u'(0)$ perpendicular to $\omega(x)$ is

$$u(t) = \frac{\sin\big(|\omega(x)|t\big)}{|\omega(x)|}\, u'(0) + \frac{\cos\big(|\omega(x)|t\big) - 1}{|\omega(x)|^2}\, \omega(x) \times u'(0).$$

Thus $\tau(x) = 2\pi/|\omega(x)|$. The rest follows from Proposition 4.1. □

In Corollary 4.2, constant length of the vorticity is necessary for the infimum of conjugate points to be monoconjugate; however, it is not sufficient, as shown by the example given in [EMP]. There, the Killing field X on the solid torus is given in cylindrical coordinates by $X = \partial_\theta$, and the vorticity is $\operatorname{curl} X = 2\,\partial_z$, with constant length. The Jacobi fields can all be computed in terms of curl eigenfields, and the monoconjugate point locations can be expressed in terms of roots of Bessel functions. The infimum of these is $\tau = \pi$, but this is not itself a monoconjugate point location. Instead, this represents an epiconjugate point; the differential of the exponential map is one-to-one, but not closed, and therefore not onto.

As we will see in the next theorem, the first conjugate point along a geodesic in $\mathcal{D}_\mu(M^3)$ is *always* pathological: the differential of the exponential map $(d\exp_{\mathrm{id}})_{\tau X_o}$ either is not closed, which implies that the span of the Jacobi fields excludes an infinite-dimensional space of vectors; or has infinite-dimensional kernel. Thus the first conjugate point is either epiconjugate of infinite order or monoconjugate of infinite order. We will present an example of this latter phenomenon later.

Theorem 4.3. *Let η be a geodesic in $\mathcal{D}_\mu(M^3)$ and let τ be the infimum of monoconjugate point locations, as in Proposition 4.1. If the differential of the exponential map $(d\exp_{id})_{\tau X_o}$ has empty or finite-dimensional kernel, then its range is not closed in the L^2 norm. Hence it is epiconjugate of infinite order, i.e., there is an infinite-dimensional space in $T_{\eta(\tau)}\mathcal{D}_\mu(M^3)$ disjoint from the image of $(d\exp_{id})_{\tau X_o}$.*

Proof. If there are no monoconjugate points, we have nothing to prove. Otherwise, τ as defined in Proposition 4.1 is finite. For each $\delta > 0$, there is some x_o with $\tau(x_o) < \tau + \delta$. Solutions of the differential equation (3.31) depend continuously on x, and thus $\tau(x)$ is a continuous function of x in a neighborhood of x_o. So for all x in some open set Ω containing x_o, we know $\tau(x) < \tau + \delta$.

In this set Ω, we can find a sequence of disjoint open sets Ψ_n, and in each one we can construct a "test" Jacobi field U_n as in Theorem 3.1, vanishing at $t = 0$ and $t = \tau + \delta$, with $I_{\tau+\delta}(U_n, U_n) < 0$. Furthermore, since the sets Ψ_n are disjoint, we have $I_{\tau+\delta}(U_n, U_m) = 0$ if $m \neq n$. Thus the space of vector fields vanishing at 0 and $\tau + \delta$ on which $I_{\tau+\delta}$ is negative-definite is infinite-dimensional. Consequently, we must have infinitely many linearly independent Jacobi fields, each of which vanishes at $t = 0$ and at some $t < \tau + \delta$, as in the proof of the finite-dimensional Morse Index theorem.

This happens one of two ways: either there are infinitely many independent Jacobi fields all satisfying $J(0) = 0$ and $J(\tau) = 0$; or there is a sequence of distinct monoconjugate point locations τ_n decreasing to τ. In the former case, we are done since $\eta(\tau)$ is then monoconjugate to $\eta(0)$ of infinite order. In the latter case, $(d\exp_{\mathrm{id}})_{\tau X_o}$ is not closed, by a result of Biliotti, Exel, Piccione, and Tausk [BEPT]. □

Remark 4.4. The result of [BEPT] is applicable in the topology generated by the Riemannian metric, i.e., the L^2 topology. One may also ask whether $(d\exp_{\mathrm{id}})_{\tau X_o}$ also has non-closed range in the Sobolev H^s topology. If M has no boundary, one can proceed as in [EMP]; the commutators of the partial derivative operators with $d\exp$ are compact, and thus one can conclude non-closed range in H^s from non-closed range in L^2. If M does have a boundary, these commutators may not be compact, so one cannot answer this question (in the same way that Fredholmness in H^s of the exponential map on $\mathcal{D}_\mu(M^2)$ is unknown if M has a boundary).

As a result of Theorem 4.3, we can say there always is a "first conjugate point," either monoconjugate or epiconjugate, found at $\eta(\tau)$, where $\tau = \inf_{x\in\mathrm{int}(M)} \tau(x)$, as in Proposition 4.1.

It is natural to ask whether the monoconjugate point may be of infinite order, and whether the first conjugate point is actually monoconjugate; the example given in [EMP] exhibits neither. To answer this question, we consider the example of $M = S^3$, where the velocity field X is a left-invariant Killing field. Misiołek [M1] showed that the corresponding geodesic in $\mathcal{D}_\mu(S^3)$ has a monoconjugate point occurring at $t = \pi$.

Using a basis of curl eigenfields on S^3, we compute all of the conjugate points in this example. They are quite interesting.

Proposition 4.5. *If $M = S^3$ and X is a left-invariant vector field, then along the corresponding geodesic η in $\mathcal{D}_\mu(S^3)$, the point $\eta(a)$ is monoconjugate to $\eta(0)$ if and only if $a = n\pi/j$ for positive integers n and j, with $j \leq n$. Each such monoconjugate point has infinite order. In addition, for any $t \geq \pi$, $d(\exp_{id})_{tX}$ does not have closed range; thus the differential of the exponential map is never Fredholm at or beyond the first conjugate point.*

Proof. Let us consider S^3 as consisting of the unit quaternions in $\mathbb{R}^4$, with coordinates (w, x, y, z). Then a basis of left-invariant vector fields is

$$\begin{aligned} e_1 &= x\,\partial_w - w\,\partial_x + z\,\partial_y - y\,\partial_z, \\ e_2 &= y\,\partial_w - z\,\partial_x - w\,\partial_y + x\,\partial_z, \\ e_3 &= z\,\partial_w + y\,\partial_x - x\,\partial_y - w\,\partial_z. \end{aligned}$$

These have unit length and satisfy the bracket relations

$$[e_1, e_2] = 2e_3, \qquad [e_2, e_3] = 2e_1, \qquad [e_3, e_1] = 2e_2.$$

Their curls satisfy

$$\operatorname{curl} e_i = -2e_i,$$

and by linearity, every vector X in the Lie algebra of S^3 also satisfies $\operatorname{curl} X = -2X$. Thus by (2.14), every such X is a steady solution of the Euler equation.

By bi-invariance of the metric on S^3, every left-invariant X is also a Killing field, so that the resulting flow consists of isometries. Thus the metric pullback is $\Lambda(t, x) \equiv 1$, and by formula (2.27), the Jacobi equation for $J = D\eta(U)$ takes the form

$$U_{tt} - 2P(X \times U_t) = 0.$$

Computing curl of both sides, and using the fact that curl annihilates gradients, we get

$$\operatorname{curl} U_{tt} + 2[X, U_t] = 0. \tag{4.40}$$

Now we have the general formula

$$\nabla\langle A, B\rangle = \nabla_A B + \nabla_B A + A \times \operatorname{curl} B + B \times \operatorname{curl} A,$$

for any vector fields A and B. Since X is a Killing field, $2\nabla_U X = \operatorname{curl} X \times U$, so we have the formula

$$\nabla\langle U, X\rangle = [X, U] + X \times \operatorname{curl} U,$$

which implies upon computing curls that $\operatorname{curl}[X, U] = [X, \operatorname{curl} U]$ for any vector field U. Thus $\mathcal{L}_X$ and curl commute as operators, so that in the (finite-dimensional) eigenspaces of curl, $\mathcal{L}_X$ restricts to an operator from each eigenspace to itself. In the L^2 metric, curl is self-adjoint while $\mathcal{L}_X$ is anti-self-adjoint. Therefore, there is a basis of the (complexified) space of divergence-free fields on S^3, orthonormal in L^2, consisting of vector fields U such that

$$\operatorname{curl} U = \lambda U \qquad \text{and} \qquad [X, U] = i\alpha U \tag{4.41}$$

for some real numbers λ and α.

To find this, we first start with the related question of the eigenvalues of X and Δ on functions. Without loss of generality, we may assume that $X = e_1$ by rotational symmetry. One can compute that the eigenfunctions of the Laplacian consist of restrictions of homogeneous, harmonic polynomials in $\mathbb{R}^4$ having degree some nonnegative integer

k; for any such f, we have $\Delta f = -k(k+2)f$. To find the eigenvalues of the operator $f \mapsto e_1(f)$, we use a toroidal coordinate system: define (σ, θ, ϕ) so that

$$w = \cos\sigma\cos\theta, \quad x = \cos\sigma\sin\theta, \quad y = \sin\sigma\cos\phi, \quad z = \sin\sigma\sin\phi.$$

Here $\sigma \in [0, \pi/2]$ while $\theta, \phi \in [0, 2\pi)$. In these coordinates, we can compute that $e_1 = \partial_\theta + \partial_\phi$. Any monomial $P = w^{k_1}x^{k_2}y^{k_3}z^{k_4}$ with $k_1 + k_2 + k_3 + k_4 = k$ can be written in these coordinates as

$$P = \cos^{k_1+k_2}\sigma \sin^{k_3+k_4}\sigma \cos^{k_1}\theta \sin^{k_2}\theta \cos^{k_3}\phi \sin^{k_4}\phi,$$

and these functions are in turn spanned by the trigonometric functions

$$p = \cos^{k_1+k_2}\sigma \sin^{k_3+k_4}\sigma e^{im_1\theta}e^{im_2\phi},$$

for some integers $m_1 \in \{k_1+k_2, k_1+k_2-2, \ldots, -(k_1+k_2)\}$ and $m_2 \in \{k_3+k_4, k_3+k_4-2, \ldots, -(k_3+k_4)\}$. For any such p, we have $e_1(p) = i(m_1+m_2)p$, so that the eigenvalue (m_1+m_2) takes on every integer value between $-k$ and k with the same odd/even parity as k.

Thus we have a basis of complex functions f such that $\Delta f = -k(k+2)f$ for some nonnegative integer k and $e_1(f) = imf$ for some integer $m \in \{-k, -k+2, \ldots, k-2, k\}$. We now construct a convenient basis of curl eigenfields, following Jason Cantarella (personal communication).

$$\begin{aligned}
&\text{(I)}: U = \operatorname{curl}^2 S + (k+2)\operatorname{curl} S \text{ for } S = fe_1 \text{ and } k \geq 2;\\
&\qquad\qquad \text{then } \operatorname{curl} U = kU \text{ and } [e_1, U] = imU.\\
&\text{(II)}: U = \operatorname{curl}^2 S + (k+2)\operatorname{curl} S \text{ for } S = f(e_2 \pm ie_3) \text{ and } k \geq 2;\\
&\qquad\qquad \text{then } \operatorname{curl} U = kU \text{ and } [e_1, U] = i(m \mp 2)U.\\
&\text{(III)}: U = \operatorname{curl}^2 S - k\operatorname{curl} S \text{ for } S = fe_1 \text{ and } k \geq 0;\\
&\qquad\qquad \text{then } \operatorname{curl} U = -(k+2)U \text{ and } [e_1, U] = imU.\\
&\text{(IV)}: U = \operatorname{curl}^2 S - k\operatorname{curl} S \text{ for } S = f(e_2 \pm ie_3) \text{ and } k \geq 0;\\
&\qquad\qquad \text{then } \operatorname{curl} U = -(k+2)U \text{ and } [e_1, U] = i(m \mp 2)U.
\end{aligned}$$

From these formulas, we see that the eigenvalues λ of curl are all integers except 0 and ± 1; for each such λ, there is an eigenvalue $i\alpha$ of $\mathcal{L}_{e_1}$, where α is an integer with $|\alpha| \leq |\lambda|$ and α having the same odd/even parity as λ. (Although it appears that type (II) violates this rule when $m = \mp k$, it turns out that U vanishes in this case.)

In such a basis, equation (4.40) is diagonalized, and the coefficient $c(t)$ of an eigenfield U with (4.41) will satisfy the equation

$$\lambda c''(t) + 2i\alpha c'(t) = 0.$$

If $\alpha \neq 0$, the solution with $c(0) = 0$ is

$$c(t) = \frac{\lambda}{\alpha}\sin\left(\frac{\alpha t}{\lambda}\right)e^{-i\alpha t/\lambda}\, c'(0),$$

and the corresponding conjugate point occurs at $t = \frac{|\lambda|\pi}{|\alpha|}$. (If $\alpha = 0$, there is no conjugate point obtained.) Since λ and α must both be integers, we have shown that all conjugate point locations are of the form $q\pi$ with $q \geq 1$ a rational number. We can obtain all such rational numbers infinitely many times; if $q = n/j$ with $j \leq n$ positive integers, then for any positive integer L, we can construct a vector field U of type (I) using $k = 2nL$ and $m = 2jL$; then the corresponding Jacobi field vanishes at $q\pi$ for any L.

The final claim, that $(d\exp_{\mathrm{id}})_{tX}$ is not closed if $t \geq \pi$, follows from the general result of [BEPT], that the differential of the exponential map is not closed at any limit point of the set of monoconjugate point locations. Since the rational multiples of π are dense in $[\pi, \infty)$, the differential of the exponential map is not closed beyond the first conjugate point. □

It would be interesting to see if a similar result is true in general; that is, whether the monoconjugate point locations are always dense in some intervals. We conjecture that they are, and expect that the proof involves a similar but more sophisticated approximation of the Jacobi field solution operator as in Theorem 3.1.

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